# Transfinite Vector Spaces

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May 2, 2007

Here we seek to discover how familiar properties of finite vector spaces are altered when the vector spaces are considered in the infinite case.

## 1 Cardinality

To best understand transfinite vector spaces, we begin with a definition.

**Definition 1 (Cardinality of a Set)** *The cardinality of a set, finite or infinite, is the number, $i$, of vectors in the set $\{u_1, u_2, u_3 ... u_i\}$.*

It is important to distinguish cardinality from dimension, which is defined as the number of elements in a spanning set. Sets with finite cardinality are discussed less often than sets with transfinite cardinality. Examples of such sets are the set spanned by the zero vector, $\langle\{\vec{0}\}\rangle$, which has card=1, and the set of single-digit natural numbers, $\mathcal{N}$, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, which has card=9.

Sets with transfinite cardinality can be said to have countably infinite cardinality or uncountably infinite cardinality. The cardinality of the set $v$ is said to be countably infinite if there exists an injection between $v$ and the set of natural numbers. It is represented by the symbol $\aleph_0$, read "aleph." An uncountably infinite set has a greater cardinality than that of the set of integers; there exists no injection between such a set and the set of integers. The real numbers are an example of such a set. Sets with uncountably infinite cardinality will not prove relavent; this discussion will concern only countably infinite sets. Now we introduce a theorem to which there has already been allusion.

**Theorem 1 (Equal Cardinality)** *Two sets, A and B have equal cardinality if there exists and injection between them. Then cardA=cardB.*

# 2 Infinite Systems of Linear Equations and Infinite Matrices

We now consider infinite systems of equations, which will be represented by matricies of infinite dimension. Such equations take the form

$$
\begin{aligned}
&\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \cdots = b_1\\
&\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \cdots = b_2\\
&\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \cdots = b_3\\
&\vdots\\
&\alpha_{m1}x_1 + \alpha_{m2}x_2 + \alpha_{m3}x_3 + \cdots = b_m
\end{aligned}
$$

or

$$
\begin{aligned}
&\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \cdots = b_1\\
&\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \cdots = b_2\\
&\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \cdots = b_3\\
&\vdots
\end{aligned}
$$

Where the former is a $m$ x $\aleph_0$ matrix and the latter is a $\aleph_0$ x $\aleph_0$ matrix. (noting here that a matrix with an infinite number of rows and a finite number of columns would have full rank when $j \geq i$.

Both matrix addition and matrix scalar multiplication behave the same way in both finite and infinite space. That is, $[A]_{ij} + [B]_{ij} = [A + B]_{ij}$ and

$\alpha[A]_{ij} = [\alpha A]_{ij}$. In addition, matrix multiplication has the same definition, $\sum_{k=1}^{n}[A]_{ik}[B]_{kj}$ with the added condition that this infinite series converges when $n$ is $\infty$. Therefore, in $\mathcal{M}_{m\aleph}$ and $\mathcal{M}_{\aleph\aleph}$ matrix multiplication is not closed because the infinte series may not converge.

The infinite $m$ x $\aleph$ matirx can be row reduced with a finite number of row operations, preserving the solution set between the original matrix and the matrix in reduced row echelon form. However, there is no guarantee that the $\aleph$ x $\aleph$ matrix can be as well. It may take infinitely many row operations, in which case it is unlikely that the solution set from the original matrix is preserved. An infinite number of terms operating on an infinite series, with possible convergence issues, does not always yeild a solution set, especially a correct one.

## 3 Existence of Matrix Products

We know well that even for $n \times n$ square matrices $A$ and $B$, $AB$ does not, in general, equal BA. However, when $n$ is finite, both products will exist. When $n$ is infinite, one product may exist while the other does not.

Consider the infinite $\aleph_0 \times \aleph_0$ matrix A in which, for every $j > 1$, $A_{ij} = 0$

$$\begin{bmatrix} a_{1j} & 0 & 0 & 0 & \cdots \\ a_{2j} & 0 & 0 & 0 & \cdots \\ a_{3j} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and an arbitrary infinite $\aleph_0 \times \aleph_0$ matrix, $B$.

Then $[AB]_{ij} = \sum_{k=1}^{\infty} a_{ik}b_{kj} = a_{i1}b_{1j}$ which must exist for any $a_{i1}$ and $b_{1j}$. Therefore, $AB$ exists. However, $[BA]_{ij} = \sum_{k=1}^{\infty} b_{ik}a_{kj}$ although when $j > 1$, $[BA]_{ij} = 0$ when $j = 1$, $[BA]_{ij} = \sum_{k=1}^{\infty} b_{ik}a_{k1}$ If this sum diverges, the product of $BA$ does not exist.

## 4 Distributivity of Infinite Matrices

The distributive property of finite matrices looks familiar:

$$A(B+C) = AB + AC \text{ or } (B+C)A = BA + CA$$

This property holds for infinite matrices, provided that $AB$ and $AC$ exist. Then $A(B+C)$ exists and is equal to $AB+AC$. However, $A(B+C)$ may exist when $AB$ and $AC$ do not. For example, let $\sum_{i=1}^{\infty} d_i$ be a convergent sum. Let $B$ be a matrix such that $b_{ij} = d_i + 1$ and let $C$ be a matrix such that $c_{ij} = d_i - 1$. Futhermore, let $A$ be a matrix such that every $a_{ij} = 1$.

$$B + C = [b]_{ij} + [c]_{ij} = [b+c]_{ij} = [d_i + 1 + d_i - 1]_{ij} = [2d_i]_{ij}$$

$A(B+C) = \sum_{k=1}^{\infty} a_{ik}[b+c]_{kj} = \sum_{k=1}^{\infty} a_{ik}[2d_k]_{kj} = \sum_{k=1}^{\infty}[1][2d_k]_{kj}$
which converges for each value of $j$ because $\sum_{i=1}^{\infty} d_i$ converges.

However, $A(B) = \sum_{k=1}^{\infty} a_{ik}b_{kj} = \sum_{k=1}^{\infty}[1]b_{kj} = \sum_{k=1}^{\infty} d_k + 1$, a sum which diverges by the $n$th-term test.

Similarly, $A(C) = \sum_{k=1}^{\infty} a_{ik}c_{kj} = \sum_{k=1}^{\infty}[1]c_{kj} = \sum_{k=1}^{\infty} d_i - 1$ which the $n$th-term test also shows to be a divergent sum. Therefore, we have infinite matrices $A, B$ and $C$ such that $A(B+C) \neq A(B) + A(C)$.

Clearly, problems with set closure arise when the distributivity and even existence of a product cannot be guaranteed. Now that it is understood that closure is reliant on convergence of matrix products, we seek matrices for which these products will always converge. Diagonal matrices, row-finite matrices and column-finite matrices are closed under the sum and product (i.e. every sum or product of diagonal, row-finite or column-finite matrices will yield a diagonal, row-finite, or column-finite matrix.)

# 5 Hamel Bases

As in finite matrix theory, we are concerned about the relation of the vectors that form the columns of such a matrix. Now we consider in what case such vectors can be said to form a basis.

**Definition 2 (Hamel Basis)** *A Hamel Basis is a linear nontopological algebraic basis which is a maximal linearly independent subset of a vector space. It has a unique dimension, called Hamel Dimension.*

It is noted that Hamel Bases exist for both finite and infinite vector spaces. In the infinite case, some of the properties of finite bases remain. Any vector in the vector space, $v$, can be written as a linear combination of vectors of the Hamel Basis. This representation is unique in the finite and infinite cases.

**Theorem 2 (Criteria for a Hamel Basis)** *Either of the following is a necessary and sufficient condition for* $b$=$\{b_1, b_2, b_3 \ldots b_n\}$ *for a linear space,* $V^n$, $b \in V^n$.
*(a) The subset b is linearly independent and every vector in* $V$ *can be written as a linear combination of vectors in* $b$.
*(b) Every vector in* $V^n$ *is a unique linear combination of vectors in* $b$.

Recall, in the finite vector space $\mathcal{C}^n$, we say that $\{c_1, c_2, c_3 \ldots c_n\}$ is a basis if either:
(a) It is linearly independent.
or
(b) Any vector in $u \in \mathcal{C}^n$ can be written as a linear combination of vectors in $c$. ($c$ spans $\mathcal{C}^n$)
However, Goldilocks is not so bold in infinite vector space, and these properties do not hold. We will show this by counter example.

Let $n$ be a transfinite cardinal, and $V^n$ the set of polynomial functions with real coefficients. Then $V$ has Hamel Dimension $\aleph_0$. Consider:

(a) The subset $\{x^2, x^4, x^6, x^8 ...\}$ which is easily shown to be linearly independent and has cardinality $\aleph_0$ but clearly does not span $V$, and therefore is not a basis.

(b) The subset $\{1, x, x^2, x^3, x^3 + x^4, x^4, x^5 \ldots x^p \ldots\}$ all terms having the form $x^p, p = 5, 6, 7 \ldots$, which has cardinality $\aleph_0$ and spans $V$. However, it is not linearly independent, so it is not a basis for $V$.

Therefore, even if a set has the correct cardinality, linear independence does not imply spanning, nor spanning imply linear independence in infinite space. Each must be checked seperately to verify a Hamel basis.

**Theorem 3 (Linear Dependence in Space with Transfinite Dimension)**

*Let $U$ be a linear space with transfinite Hamel Dimension, $\aleph_0$. Then every subset $u \in U$ with cardinality $\aleph_0 + 1$ or more is linearly dependent in $U$.*

Proof: This result follows from the ability to extend a linearly independent subset, $S$ in the vector space $\mathcal{V}$. If a vector, $w \notin < \{S\} >$, then $S \cup w = S\prime$, a linearly independent set. A Hamel basis can be created by adding each element not included in $< \{S\} >$ until no such elements remain. Because $\mathcal{V}$ is a transfinite space, the cardinality of the Hamel basis is $\aleph_0$. Because $S\prime$ is a Hamel basis, it spans $\mathcal{V}$ and is maximally linearly independent, and therefore no vector in $\mathcal{V}$ cannot be written as a linear combination of vectors in $S$. Therefore, any set with cardinality $\aleph_0 + 1$ or more is linearly dependent in $V$.

This result at first seems transparent: a maximal linearly indepenent set in $U$ has cardinality equal to the dimension of $U$. However, $\aleph_0$ and $\aleph_0 + 1$ are both countably infinite, giving us one set with countably infinite cardinality and linear independence, and another with contably infinite cardinality and linear dependence. This clarifies why spanning cannot imply linear independence, and gives meaning to infinite cardinality.

# 6 An Example of an Infinite Vector Space: Hilbert Space

**Definition 3 (Inner Product Space)** *Let $V$ be a vector space over $\mathcal{C}$. An inner product on $V$ is defined $\langle , \rangle : V\ x\ V \longrightarrow \mathcal{C}$ and has the following properties:*
*(a) For all $\vec{v} \in V, \langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = 0$. (Postive Definiteness)*
*(b) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \bar{\rangle}$ (Hermitian symmetry)*
*(c) $\langle r\vec{u} + s\vec{v}, w \rangle = r\langle \vec{u}, \vec{w} \rangle + s\langle \vec{v}, \vec{w} \rangle$ (Linearity in the first coordinate)*

Consider the infinite system with $m$ equations and $\aleph_0$ variables and define:
$\|\vec{\alpha_i}\| = \sum_{j=1}^{+\infty} |\alpha_{ij}|^2$
where $\vec{\alpha_{1i}} = (\vec{\alpha_{1i}}, \vec{\alpha_{2i}}, \vec{\alpha_{3i}}, \ldots)$.
When this series converges, $\|\vec{\alpha_i}\| = \infty, \vec{\alpha_i}$ has a finite norm.

**Definition 4 (Hilbert Space)** *The set of all $\vec{\alpha_i}$ with a finite norm make up an inner product space, Hilbert space, $\ell_2$.*

Thus, Hilbert space is a carefully defined linear space with cardinality $\aleph_0$.

**Theorem 4 (Linear Dependence in Hilbert Space)** *The set of vectors $\vec{\alpha_i} \in \ell_2$ is linearly dependent if and only if the inner product of any two vectors in the set, or of a vector with itself, is zero.*

Finally, to conclude, we offer a theorem that guarantees the existence of that which we always seek when presented with an initial system of equations: a solution. Recalling the initial system of equations,

**Theorem 5 (Existence of a solution in Hilbert Space)** *If $\vec{\alpha_i}, i = 1, 2, 3, ...m \in \ell_2$ is a linearly independent set, there exists a unique linear combination of $\vec{\alpha_i}$ which equals $\vec{b}$. $[\alpha_1|\alpha_2|\alpha_3|\cdots|\alpha_m]\vec{x} = \vec{b}$.*

This, again, does not sound so impressive until we realize that $\vec{x}$ has dimension $\aleph_0$, that is, it represents a countably infinite number of scalars multiplying a countably infinite number of vectors to create a linear combination equalling a vector, $\vec{b}$ from $\mathcal{C}^m$, with only a few restrictions on convergence and inner products.

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