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Math 420: Advanced Linear Algebra
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Bilinear and Quadratic Forms

This paper will concentrate on bilinear and quadratic forms and some of the unique properties that go along with them. I will start by presenting bilinear forms and how they generalize some concepts that we have already encountered and then proceed to some of the more interesting theorems behind bilinear forms. I will then introduce quadratic forms and how linear algebra and calculus connect with eachother.

Section: Bilinear Forms

Bilinear Forms: If V is a vector space over the scalar field $\mathbf{F}$, a $bilinear form$ is a function, $f : V \times V \to \mathbf{F}$,which satisfies the following conditions:

(1) $f(u+v,w) = f(u,w)+f(v,w), f(u,v+w) = f(u,v)+f(u,w)$, and $f(kv,w) = kf(v,w) = f(v,kw)$ for all $u, v, w \in V$ and all $k \in \mathbf{F}$.

(2) f is called a *bilinear symmetric form* if it also satisfies:
$f(v,w) = f(w,v)$ for all $v, w \in V$.

Something to notice about the definition of a bilinear form is the similarity it has to an inner product. In essence, a bilinear form is a generalization of an inner product! Now that we know what a bilinear form is, here is an example.

Example: Suppose that A is an $n \times n$ matrix. For $u, v \in F^n$ we will define the function

$$f(u,v) = u^t Av \in \mathbf{F}$$

Lets check then if this is a bilinear form. $f(u+v,w) = (u+v)^t Aw = (u^t+v^t)Aw = u^t Aw+v^t Aw = f(u,w)+f(v,w)$. Also, $f(\alpha u,v) = (\alpha u)^t Av = \alpha(u^t Av) = \alpha f(u,v)$. We can see then that our defined function is bilinear. Looking at how this function is defined, especially the matrix A, it might give us a hint to a similarity between this bilinear form and the linear transformations we have been working with. The next theorem will make the relation a bit more apparent.

Theorem BFLP: Bilinear Forms have Linear Parts

Suppose that $f : V \times V \to \mathbf{F}$ is a bilinear form. For a $v \in V$, the functions P_v and $R_v : V \to \mathbf{F}$ defined by $P_v(u) = (v,u)$ and $R_v(u) = (u,v)$ are linear transformations.

Proof: $P_v(u+w) = (v,u+w) = (v,u)+(v,w) = P_v(u)+P_v(w)$ and $P_v(\alpha u) = (v,\alpha u) = \alpha(v,u) = \alpha P_v(u)$. So we know that P_v is linear. How about R_v? $R_v(u+w) = (u+w,v) = (u,v)+(w,v) = R_v(u)+R_v(w)$ and $R_v(\alpha u) = (\alpha u,v) = \alpha(u,v) = \alpha R_v(u)$. We can see then that bilinear forms are linear in each component. If bilinear forms are linear in each component and the example of a bilinear form had a matrix involved...hmmmmmm. The next theorem will then let us safely say that EVERY bilinear form defined as $f : \mathbf{F}^m \times \mathbf{F}^n \to \mathbf{F}$ has a unique matrix such that $f(x,y) = x^t Ay$ for all $x \in F^m$ and $y \in F^n$.

Theorem BFCUM: Bilinear Forms of Column vectors have Unique Matrices

Given a bilinear form $f : \mathbf{F}^m \times \mathbf{F}^n \to \mathbf{F}$, there exists a unique matrix, $A_{m\times n}$ such that $f(x, y) = x^t Ay$ for all $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$.

Proof: We will use the standard basis vectors to define $\mathbf{F}^m$ and $\mathbf{F}^n$. We know then that for the vector $x \in \mathbf{F}^m$ and the vector $y \in \mathbf{F}^n$, we can represent these vectors as $x = \sum_{i=1}^m [x]_i e_i$ and $y = \sum_{j=1}^n [y]_j e_j$. Using the properties of bilinear forms then

$$f(x, y) = f(\sum_{i=1}^m [x]_i e_i, \sum_{j=1}^n [y]_j e_j) =$$

$$\sum_{i=1}^m \sum_{j=1}^n [x]_i [y]_j f(e_i, e_j).$$

We will now define the matrix A as $[A]_{ij} = f(e_i, e_j)$. We can see then that $f(x, y)$ becomes

$$f(x, y) = \sum_{i=1}^m \sum_{j=1}^n [x]_i [A]_{i,j} [y]_j = x^t Ay.$$

To prove the uniqueness of the matrix A, we will assume that there is another matrix B that also satisfies the bi-linearity of $f(x, y)$. $f(x, y) = x^t Ay = x^t By$.

$$x^t Ay = x^t By \Rightarrow x^t Ay - x^t By = x^t(A - B)y = 0$$

If we set let the matrix $C = A - B$ we can define our expression as $\sum_{i=1}^m \sum_{j=1}^n [x]_i [C]_{i,j} [y]_j = 0$. If we choose any $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$, then we find that $[C]_{i,j} = 0$ for all i and j. Then, $C = A - B = 0 = A = B$! We have shown then that the matrix A is unique.

Example: Given the bilinear form $f : \mathbf{F}^3 \to \mathbf{F}^3$ defined by $f((\alpha, \beta, \gamma), (\delta, \epsilon, \theta)) = \alpha(\epsilon + \theta) - 2\gamma\delta$, we can see that by computing with the standard basis for $\mathbf{F}^3$, we get the matrix entries

$$\begin{gathered}
[A]_{1,1} = ((1,0,0),(1,0,0)) = 1(0+0) - 2(0)(0) = 0 \\
[A]_{1,2} = ((1,0,0),(0,1,0)) = 1(1+0) - 2(0)(0) = 1 \\
[A]_{1,3} = ((1,0,0),(0,0,1)) = 1(0+1) - 2(0)(0) = 1 \\
[A]_{2,1} = ((0,1,0),(1,0,0)) = 0(0+0) - 2(0)(1) = 0 \\
[A]_{2,2} = ((0,1,0),(0,1,0)) = 0(1+0) - 2(0)(0) = 0 \\
[A]_{2,3} = ((0,1,0),(0,0,1)) = 0(0+1) - 2(0)(0) = 0 \\
[A]_{3,1} = ((0,0,1),(1,0,0)) = 0(0+0) - 2(1)(1) = -2 \\
[A]_{3,2} = ((0,0,1),(0,1,0)) = 0(1+0) - 2(1)(0) = 0 \\
[A]_{3,3} = ((0,0,1),(0,0,1)) = 0(0+1) - 2(1)(0) = 0
\end{gathered}$$

$$[\alpha, \beta, \gamma] \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \epsilon \\ \theta \end{bmatrix} = \begin{bmatrix} \alpha\epsilon + \alpha\theta - 2\gamma\delta \end{bmatrix} \mathit{BOOM!}$$

Section: Quadratic Forms

We could go on to define other interesting properties of bilinear forms but defining bilinear forms was only necessary to introduce *quadratic forms*. What this section is going to demonstrate is that it is possible to take symmetric bilinear forms and find analogous quadratic forms. Like always, we will begin with a definition.

Quadratic Forms: Let V be a vector space over the field $\mathbf{F}$. A quadratic form is a funtion $f : V \to F$ such that the following hold.

(1) $f(kv) = k^2 f(v)$ for all $v \in V$ and $k \in F$.

(2) $b_f(u,v) = f(u+v) - f(u) - f(v)$ is a symmetric bilinear form.

N.B. Given a quadratic form, the notation b_f denotes the packaged symmetric bilinear form. In practice, $(1/2)b_f$ is considered the bilinear form associated with f. The factor of $(1/2)$ appears to come with no motivation, but the next theorem will show why it's there. It is also from this theorem that we see the geometrical implications of a particular quadratic form.

Theorem SBFQF: Symmetric Bilinear Forms have Quadratic Forms

If f is a symmetric bilinear form in V, then $f(v) = (v,v)$ is a quadratic form in V. Also, $b_f(u,v) = 2(u,v)$ for all $u, v \in V$.

Proof:

$$\begin{aligned}
(1) f(kv) &= (kv, kv) = k^2(v,v) = k^2 f(v). \\
(2) b_f(u,v) &= f(u+v) - f(u) - f(v) \\
&= (u+v, u+v) - (u,u) - (v,v) \\
&= (u, u+v) + (v, u+v) - (u,u) - (v,v) \\
&= (u,u,) + (u,v) + (v,u) + (v,v) - (u,u) - (v,v) = 2(u,v).
\end{aligned}$$

Since f was defined to be a symmetric bilinear form, we can see that b_f is a symmetric bilinear form.

What happens when $(,)$ is defined as an inner product over the real numbers? Lets check shall we? Let $f(v) = (v,v)$ be a quadratic funtion defined with the inner product.

$$b_f(v,v) = f(v+v) - f(v) - f(v)$$
$$= (v+v, v+v) - (v,v) - (v,v)$$
$$= (v, v+v) + (v+v, v) - 2(v,v)$$
$$= 4(v,v) - 2(v,v)$$
$$= 2(v,v)$$
$$(1/2)b_f(v,v) = f(v,v) = \sum_{k=1}^{n} v_k^2 = ||v||^2$$

So, $(1/2)b_f(v,v)$ is the associated bilinear form of the quadratic form $f(v) = (v,v)$ when $(,)$ is defined as an inner product, and we can see that $f(v) = (v,v)$ is equal to the square of the vector norm. So where does that leave us? We could talk more about the geometrical topics of the quadratic form but that's for differential geometry. What is next is even more amazing than showing there is a matrix associated with bilinear forms.

Theorem QFFQP: Quadratic Forms come from Quadratic Polynomials

(1)For any homogeneous quadratic polynomial defined as $p(x_1, \cdots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}x_ix_j$ where $a_{i,j} \in \mathbf{F}$, let S_p be the symmetric matrix with entries $[S_p]_{i,j} = (1/2)(a_{i,j} + a_{j,i})$. Then, for all $v = (\phi_1, \cdots, \phi_n) \in \mathbf{F}^n$, $p(\phi_1, \cdots, \phi_n) = (v, v)_{S_p}$. Also, the function $f(v) = p(\phi_1, \cdots, \phi_n)$ is a quadratic form.

Proof:

$$\begin{aligned}
p(\phi_1, \cdots, \phi_n) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\phi_i\phi_j \\
&= \sum_{i=1}^n a_{ii}\phi_i^2 + \sum_{i<j}^n (a_{ij} + a_{ji})\phi_i\phi_j \\
&= \sum_{i=1}^n [S_p]_{ii}\phi_i^2 + \sum_{i<j}^n 2[S_p]_{ij}\phi_i\phi_j \\
&= \sum_{i=1}^n \sum_{j=1}^n [S_p]_{i,j}\phi_i\phi_j = \sum_{i=1}^n \phi_i \left(\sum_{j=1}^n [S_p]_{i,j}\phi_j \right) \\
&= \sum_{i=1}^n \phi_i S_p \\
\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} &= [\phi_1, \cdots, \phi_n] S_p \\
\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} &= (v, v)_{S_p}
\end{aligned}$$

To show that $f(v)$ is a quadratic form, we show the two definded properties of quadratic forms. $f(kv) = k^2 f(v)$ because our quadratic was defined as homogeneous. Also, by Theorem SBFQF, we know that we also have our needed bilinear form. Let's now show an example.

Example: Find the quadratic form associated with the symmetric matrix

$$X = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 0 & 2 \\ 5 & 2 & 9 \end{bmatrix}$$

The symmetric bilinear form associated with X is

$$f((a, b, c), (d, e, f))_X = [a, b, c] \begin{bmatrix} 1 & 4 & 5 \\ 4 & 0 & 2 \\ 5 & 2 & 9 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$= ad + 4ae + 5af + 4be + 2bf + 5cd + 2ce + 9cf$$

The quadratic form associated with X is then going to be

$$f(j,k,l) = ((j,k,l),(j,k,l))_X = j^2 + 8jk + 10jl + 4lk + 9l^2$$

which is a funtion defined by a homogeneous quadratic polynomial in three variables. To combine all the ideas that have been presented, we will concentrate on quadratic forms over the field $\mathbb{R}^n$. If we remember the second derivative test from multivariate calculus, we remember we get the symmetric matrix

$$S = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a_1, \cdots, a_n) \right)$$

that determines if critical points are minimums, maximums, or saddle points. We could use what we know about positive semi-definite matrices, but since we have just learned how to go between symmetric matrices and quadratic forms, we might as well study this matrix using them. If we were to make the matrix S of some n-dimensional function, all we would need to do is to take the quadratic form of that matrix and study where it is positive or negative. For the quadratic form, q, we can find minimum, maximum, and saddle points depending on how it behaves. If q is negative definite, for points other than the critical point, then the critical point is a local maximum. If q is positive definite then the point is a minimum. Lastly, if q is a function in which there exists points such that q can be positive or negative and the associated matrix is nonsingular, then the critical point is a saddle point. How about we crank out some examples then!

Example: Minimum Let $f(x,y) = x^2 + y^2$. If we take each partial derivative we find that we get the symmetric matrix

$$S = \begin{bmatrix} \frac{\partial f}{\partial x \partial x} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial x \partial y} & \frac{\partial f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

If we check the partials of f, we find the the only critical point is $(0,0)$, so the quadratic form defined as a homogeneous polynomial for this matrix is $q(x,y) = 2x^2 + 2y^2$.

```
In[25]:=  Plot3D[2 x^2 + 2 y^2, {x, -10, 10}, {y, -10, 10}]
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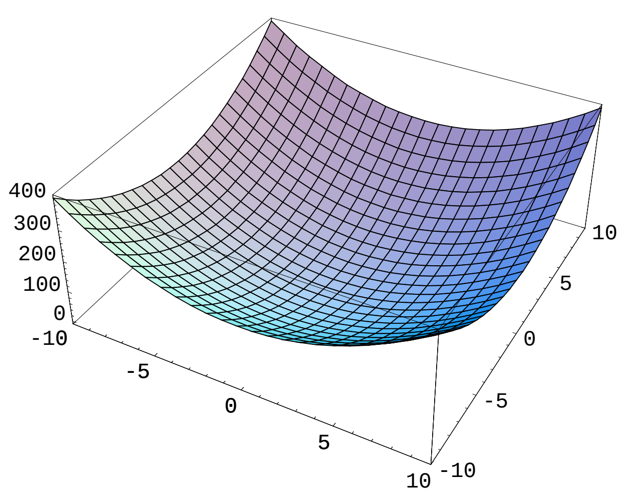


```
Out[25]=  - SurfaceGraphics -
```

We can see from the plot and the function that the q is positive definite for all values other than $(0,0)$, which means the point is a minimum. How about a maximum?

Example: Maximum Let $f(x,y) = 2 - x^2 - y^2$. Doing the same procedure we find the symmetric matrix to be

$$S = \begin{bmatrix} \frac{\partial f}{\partial x \partial x} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial x \partial y} & \frac{\partial f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

From the partials we see the only critical point to be again $(0,0)$ and the associated quadratic form is $q(x,y) = -2x^2 - 2y^2$.

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In[26]:=  Plot3D[-2 x^2 - 2 y^2, {x, -10, 10}, {y, -10, 10}]
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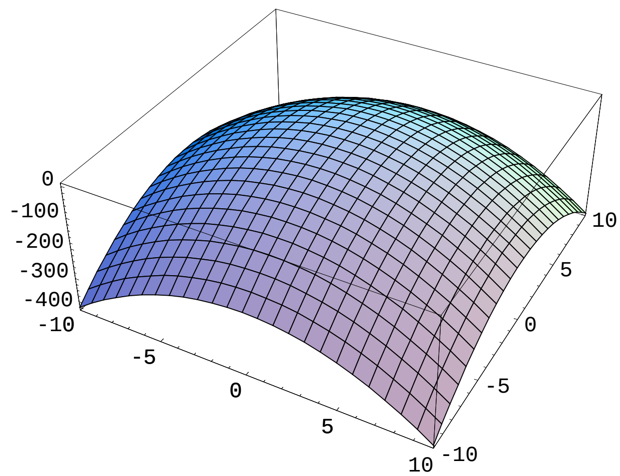


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Out[26]=  - SurfaceGraphics -
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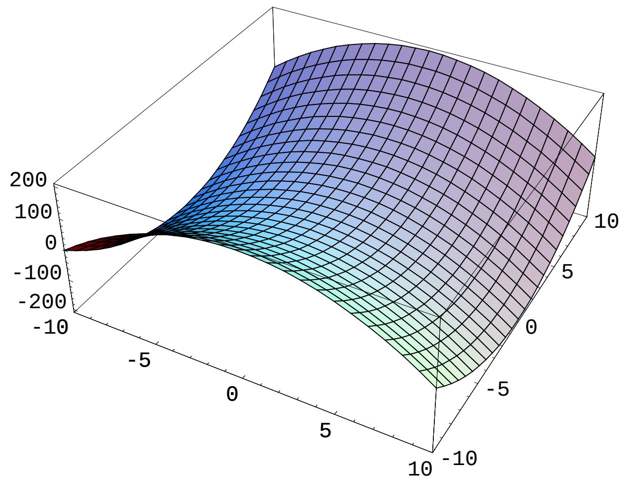
We can tell from the plot and the function that all points, other than the critical point, is less than the function evaluated at $(0,0)$. Lastly, we will look at a saddle point.

Example: Saddle Point Let $f(x,y) = y^2 - x^2$. Once again we will do the same procedure to find the symmetric matrix to be

$$S = \begin{bmatrix} \frac{\partial f}{\partial x \partial x} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial x \partial y} & \frac{\partial f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

From the partials we see the only critical point to be again $(0,0)$, and the associated quadratic form is $q(x,y) = -2x^2 + 2y^2$.

```
In[1]:=  Plot3D[-2 x^2 + 2 y^2, {x, -10, 10}, {y, -10, 10}]
```

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Out[1]=  - SurfaceGraphics -
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We can see from the plot and the function that all the points, other than the critical point, allow the function to be positive and negative. Something else to make sure of though is that our symmetric matrix is nonsingular, which it is.

N.B. The three examples that were shown were functions that had only one critical point. In most cases, the function will have multiple critical points and the symmetric matrix generated from the partial derivates of the function will need to be evaluated at each critical point and then turned into the associated quadratic form.

Conclusions: So what do bilinear, and quadratic forms do for us? First off, and most importantly, they give us a bridge between linear algebra and calculus. Linear algebra can sometimes feel too abstract compared to analysis, but quadratic forms give us an example to how we can represent particular matrices as polynomials. Secondly, bilinear and quadratic forms are the basis for multilinear algebra and the foundations of tensor analysis. Did you know that the definition of the determinant can be looked at as a multilinear function on the columns of a matrix? Also, many ideas in advanced calculus, such as differential forms and manifolds, can be understood through multilinear algebra. This paper has hopefully given a brief introduction to bilinear and quadratic forms and shown the linear algebra that is hiding in the background of analysis.

References

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