Categories of Groups and Rings: A Brief Introduction to Category Theory for Students of Abstract Algebra

Nathan Lander

April 10, 2012

As students of Abstract Algebra, we have been exposed to multiple new algebraic structures that generalize many of the ideas we have learned previously in our mathematical educations. Categories are another algebraic structure that seeks to generalize many different areas in mathematics and provide a means for studying relationships between them. Just as we have been able to prove theorems that apply to many different kinds of mathematical objects through the study of Groups and Rings, the goal of Category Theory is to prove results about Groups and Rings themselves, as well as other fields of Mathematical study.

Just as Groups and Rings have a certain axiomatic structure, Categories have their own axomatic structure. A Category is composed of a collection of **objects**, and a collection of **arrows**. The collection of objects in a category

$\mathcal{C}$ will be denoted $\mathcal{C}_0$, and the collection of arrows in $\mathcal{C}$ will be denoted $\mathcal{C}_1$. The objects and arrows of a category meet the following criteria [1], [2]:

- Each arrow points from one object, known as the arrow's **domain**, to another object, known as the arrow's **codomain**. An arrow f with an object A as its domain and an object B as its codomain will be denoted $A \xrightarrow{f} B$, when not denoted simply as f.

- For any two arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, the composition of f and g $A \xrightarrow{g \circ f} C$ is defined such that $g \circ f$ is also an arrow in $\mathcal{C}$.

- For every object A, there exists an identity arrow $A \xrightarrow{1_A} A$.

- If $A \xrightarrow{f} B$, then $f \circ 1_A = f$ and $1_B \circ f = f$.

- Composition of arrows is associative, i.e., for $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, and $C \xrightarrow{h} D$, $(h \circ g) \circ f = h \circ (g \circ f)$.

There are a few important things to note before continuing. First, arrow composition is only defined if the codomain of the first arrow is the domain of the second arrow. Also, the order of arrow composition is written from *right to left*, with the first arrow on the right. As a result, we should read $f \circ g$ as "f following g."[1]

Now, for our first example, let's apply the concept of a Category to an algebraic structure with which we are all familiar: Groups. There is a Category commonly referred to as Grp with groups as the objects and group homomorphisms as the arrows.[2]

Proof.   • In Group Theory, we have also used the words *domain* and *codomain* in relation to group homomorphisms, to refer to the group *from* which a homomorphism takes group elements and the group *to* which a homomorphism takes group elements, respectively. We will link these notions of domain and codomain in the natural way to the domains and codomains of the arrows in Grp. Thus, every arrow in Grp has a domain and a codomain.

• If ϕ is a group homomorphism from $(G, \cdot)$ to $(I, *)$, and ψ is a group homomorphism from I to $(J, \star)$, then for $g, h \in G, \psi \circ \phi(g \cdot h) = \psi(\phi(g \cdot h)) = \psi(\phi(g) * \phi(h)) = \psi(\phi(g)) \star \psi(\phi(h)) = \psi \circ \phi(g) \star \psi \circ \phi(h)$. Thus, the composition of two homomorphisms is itself a homomorphism, so we can define the composition of two arrows in Grp as the usual composition of group homomorphisms.

• The identity function in a group $(G, \cdot)$ is a group homomorphism, since for $g, h \in G, id_G(g \cdot h) = g \cdot h = id_G(g) \cdot id_G(h)$. Thus, we can let the identity function id_G of a group G be the identity arrow 1_G for the object G in Grp.

• Given a group homomorphism ϕ from G to H, for $g \in G, id_H \circ \phi(g) = id_H(\phi(g)) = \phi(g)$, and $\phi \circ id_G(g) = \phi(id_G(g)) = \phi(g)$. Thus, our identity arrows 1_H and 1_G compose appropriately with arrows in Grp.

• Consider three group homomorphisms δ from C to D, ψ from B to C, and ϕ from A to B. For $g \in G, \delta \circ (\psi \circ \phi)(g) = \delta(\psi \circ \phi(g)) =$

3

$\delta(\psi(\phi(g))) = \delta \circ \psi(\phi(g)) = (\delta \circ \psi) \circ \phi(g)$. Thus, composition of arrows in Grp is associative.

As Grp meets the criteria of a category, we can conclude that Grp is, in fact, a category. □

There is also a category commonly known as Rng, that has rings as its objects and ring homomorphisms as its arrows.[2] The proof that Rng is a category is entirely similar to the proof that Grp is a category.

Now let's define two important properties that arrows can have in a category. An arrow $A \xrightarrow{f} B$ is **mono** if for any two arrows g and h with domain C and codomain A, $f \circ g = f \circ h$ implies that $g = h$. In other words, f is mono if you can "cancel" f from any equation of two arrow compositions where f is the second arrow in both compositions (remember that the second arrow in a composition is found on the left). The second important property of arrows is a sort of converse to the property mono. An arrow $A \xrightarrow{f} B$ is **epi** if for any two arrows g and h with domain B and codomain C, $g \circ f = h \circ f$ implies that $g = h$. So f is essentialy epi if you can "cancel" f from any equation of two arrow compositions where f is the first arrow (the one on the right) in both compositions.

In Rng, if an arrow is mono, then it is injective.

Proof. Consider a mono arrow $A \xrightarrow{f} B$. Suppose for the purposes of contradiction that f is not injective. Now consider functions g and h from the kernel of f, an ideal in A, to A, defined such that g takes every element of the kernel of f to itself in A, and h takes every element of the kernel of f to 0_A.[3]

4

As, $g(k_1 + k_2) = k_1 + k_2 = g(k_1) + g(k_2)$, and $g(k_1 \cdot k_2) = k_1 \cdot k_2 = g(k_1) \cdot g(k_2)$ for k_1, k_2 in the kernel of f, g is a ring homomorphism, and hence an arrow in Rng. Also, as $h(k_1 + k_2) = 0_A = 0_A + 0_A = h(k_1) + h(k_2)$, and $h(k_1 \cdot k_2) = 0_A = 0_A \cdot 0_A = h(k_1) \cdot h(k_2)$ for k_1, k_2 in the kernel of f, h is a ring homomorphism, and hence an arrow in Rng. $f \circ g = f \circ h$ is the ring homomorphism that takes elements of the kernel of f to 0_B. However, as f is not injective, $g \neq h$, because the kernel of f is a nontrivial ideal in A. This contradicts our hypothesis that f is a monomorphism. Thus, it must be the case that f is injective.[4] $\qquad\square$

The proof that arrows in Grp are injective if they are mono is entirely similar to the proof above, but normal subgroup is used in the place of ideal. It is also true that an arrow in Grp is surjective if it is epi. It is not, however, true that an arrow in Rng is surjective if it is epi. For example, take the ring homomorphsm that takes integers to the same value, but as a rational number. This arrow in Rng is epi, but it is certainly not surjective.

In Category Theory, there is a concept that is essentially for categories what functions are for sets, and what homomorphisms are for groups and rings. This concept is that of a **functor**. A functor points from a category $\mathcal{C}$ to another category $\mathcal{D}$, and consists of two operrations F_0 and F_1 that take objects from $\mathcal{C}_0$ to $\mathcal{D}_0$ and arrows from $\mathcal{C}_1$ to $\mathcal{D}_1$ respectively. The operations F_0 and F_1 of a functor F have the following properties: [2]

- For an arrow $A \xrightarrow{f} B$ in $\mathcal{C}_1$, it must be the case that $F_1(f)$ in $\mathcal{D}_1$ has for domain $F_0(A)$ in $\mathcal{D}_0$ and for codomain $F_0(B)$ in $\mathcal{D}_0$, or $F_0(A) \xrightarrow{F_1(f)}$

$F_0(B)$.

- For $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathcal{C}_1$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

- For A in $\mathcal{C}_0$, $F_1(1_A) = 1_{F_0(A)}$.

One very simple example of a functor is the functor that takes rings in Rng to the abelian groups in Grp formed under addition and takes ring homomorphisms to group homomorphisms that preserve the opperation of addition in the two abelian groups. This functor and others like it are known as "forgetful", because they "forget" the structure of the first category.

Proof.
- The first criterion of a functor is satisfied almost trivially in this example. We have defined F_0 such that rings get sent to the same set, but with only addition defined on the set elements instead of both addition and multiplication. Similarly, we have defined F_1 such that ring homomorphisms get sent to functions of the same two sets, but that only preserve addition, instead of preserving both addition and multiplication. Therefore, as the domain and codomain of a homomorphism are essentially the same sets before and after the functor is applied, the first criteria is satisfied.

- The second criterion of a functor is satisfied here for essentially the same reason as the first. A homomorphism f in Rng_1 is essentially the same function as $F_1(f)$ in Grp_1 because each homomorphism sends the same set elements in its domain to the same set elements in its codomain. Thus, it follows that the composition $f \circ g$ of two ring

homomorphisms is essentially the same function as the composition of two group homomorphisms $F_1(f) \circ F_1(g)$.

- This functor sends the identity homomorphism of a ring to the identity homomorphism of that ring's abelian group under addition in the ring. As this functor meets the criteria of a functor, it is, in fact, a functor. □

Notice that there is no mention in the definition of a functor of a need for every element of the first category to be sent to some element of the second category. Thus, we can define another functor from Rng to Grp that takes finite fields to their multiplicative groups, and homomorphisms of finite fields to group homomorphisms such that a homomorphism of a finite field gets sent to a group homomorphism that treats all of the set items the same, but is no longer defined for the zero element of the domain, since there no longer is a zero element in the domain.

We can see already from these few basic developments that Category Theory is a broad topic that has much potential for applicability accross many areas of mathematics. I hope that by exposing you to a few of the basic terms and concepts of Category Theory I have given you a glimpse of how the concepts we have studied in Abstract Algebra are related to each other, and possibly, to many other fields of study in mathematics.

References

[1] F. W. Lawvere, S. H. Schanuel, *Conceptual Mathematics: A first Introduction to Categories*, Cambridge Univesity Press, Cambridge, (1997)

[2] J. V. Oosten, *Basic Category Theory*, `http://www.staff.science.uu.nl/~ooste110/syllabi/catsmoeder.pdf`, (2012/04/09)

[3] T. Judson, *Abstract Algebra: Theory and Applications* http://abstract.pugetsound.edu/, GFDL License, (1997)

[4] (Multiple anonymous authors), *Monomorphism in category of groups equals injective homomorphism*, `http://groupprops.subwiki.org/wiki/Monomorphism_in_category_of_groups_equals_injective_homomorphism`, (2012/04/09)