Math 290 Name: Key Dr. Beezer
Exam 1 Fall 2013
Chapter SLE

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers.

1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} 4x_1 - 3x_2 + 8x_3 &= -3 \\ -x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 4x_2 + 8x_3 &= -4 \\ 2x_1 + x_2 + 4x_3 &= 2 \end{aligned}$$

The augmented matrix of the system: $\left[\begin{array}{ccc|c} 4 & -3 & 8 & -3 \\ -1 & 1 & -2 & 1 \\ 4 & -4 & 8 & -4 \\ 2 & 1 & 4 & 2 \end{array}\right]$

"row-reduce" $\xrightarrow{\text{RREF}}$ $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The last column is a pivot column so by Theorem RCLS, the system is inconsistent.

2. Solve the following system of linear equations and express the solutions as a set of column vectors. (20 points)

$$\begin{aligned} x_1 + x_2 + 3x_4 + 2x_5 + 4x_6 &= 16 \\ x_2 - 2x_3 + 3x_4 + x_5 + 3x_6 &= 4 \\ x_1 + 2x_3 + x_5 + x_6 &= 12 \\ -x_1 + x_2 - 4x_3 + 3x_4 + 2x_6 &= -8 \end{aligned}$$

Augmented matrix of the system $\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 3 & 2 & 4 & 16 \\ 0 & 1 & -2 & 3 & 1 & 3 & 4 \\ 1 & 0 & 2 & 0 & 1 & 1 & 12 \\ -1 & 1 & -4 & 3 & 0 & 2 & -8 \end{array}\right]$

"row-reduce" $\xrightarrow{\text{RREF}}$ $\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 2 & 0 & 1 & 1 & 12 \\ 0 & \boxed{1} & -2 & 3 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -2x_3 - x_5 - x_6 + 12$
$x_2 = 2x_3 - 3x_4 - x_5 - 3x_6 + 4$

$$S = \left\{ \left. \begin{bmatrix} -2x_3 - x_5 - x_6 + 12 \\ 2x_3 - 3x_4 - x_5 - 3x_6 + 4 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \right| x_3, x_4, x_5, x_6 \in \mathbb{C} \right\}$$

3. Determine if the matrix below is nonsingular or singular. Explain your reasoning carefully and thoroughly. (15 points)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 2 & -3 \\ -1 & 1 & 1 & 0 & 0 & 2 & 3 & -3 \\ 1 & 0 & -5 & 1 & -2 & 3 & -3 & 5 \\ -1 & 0 & 5 & 0 & 1 & 0 & 2 & -4 \\ -1 & 0 & 5 & 1 & 0 & 3 & 1 & -3 \\ 1 & 0 & -5 & -1 & 3 & -6 & 5 & -6 \\ 1 & 0 & -5 & -1 & 0 & -3 & -1 & 3 \\ -2 & 1 & 6 & -1 & 0 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}}$$

A matrix that is <u>not</u> the size 8 identity matrix

By theorem NMRRI this matrix is singular.

4. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (20 points)

$$A = \begin{bmatrix} 1 & 1 & -1 & -6 & 2 \\ -1 & 0 & -2 & 4 & -2 \\ -2 & 0 & -4 & 9 & -3 \end{bmatrix} \xrightarrow[2R_1+R_3]{R_1+R_2} \begin{bmatrix} \boxed{1} & 1 & -1 & -6 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 2 & -6 & -3 & 1 \end{bmatrix} \xrightarrow[-2R_2+R_3]{-R_2+R_1} \begin{bmatrix} \boxed{1} & 0 & 2 & -4 & 2 \\ 0 & \boxed{1} & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[4R_3+R_1]{2R_3+R_2} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 6 \\ 0 & \boxed{1} & -3 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix} = B$$

5. Find all of the values of α for which the following system has a unique solution. (15 points)

$$x_1 - 3x_2 - x_3 = 3$$
$$-2x_1 + x_2 - 3x_3 = -1$$
$$x_1 + 2x_2 + \alpha x_3 = -1$$

Row-reduce, by hand, the augmented matrix of the system, carrying α as a symbol

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 3 \\ -2 & 1 & -3 & -1 \\ 1 & 2 & \alpha & -1 \end{array}\right] \xrightarrow[-R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|c} \boxed{1} & -3 & -1 & 3 \\ 0 & -5 & -5 & 5 \\ 0 & 5 & \alpha+1 & -4 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} \boxed{1} & -3 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 5 & \alpha+1 & -4 \end{array}\right]$$

$$\xrightarrow[-5R_2+R_3]{3R_2+R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & \alpha-4 & 1 \end{array}\right]$$

① $\alpha = 4 \Rightarrow$ last column will become a pivot column
$\Rightarrow$ by RCLS, system is inconsistent

② $\alpha \neq 4 \Rightarrow$ column 3 is a pivot column, $r=3$
$\Rightarrow n-r = 3-3 = 0$ so by FVCS, unique solution

6. Suppose that: (1) A is an $n \times n$ square matrix, (2) $\mathbf{b}$ is a vector with n entries and (3) $\mathcal{LS}(A, \mathbf{b})$ has a unique solution. Prove that A is nonsingular. (15 points)

Read Solution NM.T31

Outline:
1) $[A|\underset{\sim}{b}] \to [B|\underset{\sim}{c}]$
2) $\mathcal{LS}(A,\underset{\sim}{b})$ consistent, RCLS $\Rightarrow \underset{\sim}{c}$ not pivot column
3) Consistent, FVCS, unique solution $\Rightarrow n-r=0$
$\Rightarrow n=r$
4) B is all pivot columns, ie $B = I_n$
5) $A \xrightarrow{\text{rref}} I_n$, NMRRI $\Rightarrow A$ nonsingular