

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.

You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers.

1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$4x_1 - 3x_2 + 8x_3 = -3$$

$$-x_1 + x_2 - 2x_3 = 1$$

$$4x_1 - 4x_2 + 8x_3 = -4$$

$$2x_1 + x_2 + 4x_3 = 2$$

The augmented matrix of the system:
$$\left[\begin{array}{ccc|c} 4 & -3 & 8 & -3 \\ -1 & 1 & -2 & 1 \\ 4 & -4 & 8 & -4 \\ 2 & 1 & 4 & 2 \end{array} \right]$$

"row-reduce"

RREF



$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The last column is a pivot column so by Theorem RCLS, the system is inconsistent.

2. Solve the following system of linear equations and express the solutions as a set of column vectors. (20 points)

$$x_1 + x_2 + 3x_4 + 2x_5 + 4x_6 = 16$$

$$x_2 - 2x_3 + 3x_4 + x_5 + 3x_6 = 4$$

$$x_1 + 2x_3 + x_5 + x_6 = 12$$

$$-x_1 + x_2 - 4x_3 + 3x_4 + 2x_6 = -8$$

Augmented matrix of the system

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 3 & 2 & 4 & 16 \\ 0 & 1 & -2 & 3 & 1 & 3 & 4 \\ 1 & 0 & 2 & 0 & 1 & 1 & 12 \\ -1 & 1 & -4 & 3 & 0 & 2 & -8 \end{array} \right]$$

"row-reduce"

RREF

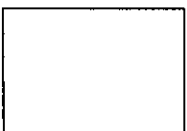


$$\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 0 & 1 & 1 & 12 \\ 0 & 1 & -2 & 3 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 = -2x_3 - x_5 - x_6 + 12$$

$$x_2 = 2x_3 + 3x_4 - x_5 - 3x_6 + 4$$

$$S = \left\{ \begin{array}{l} \left[\begin{array}{cccc|c} -2x_3 & -x_5 - x_6 + 12 \\ 2x_3 - 3x_4 - x_5 - 3x_6 + 4 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right] \mid x_3, x_4, x_5, x_6 \in \mathbb{C} \end{array} \right\}$$



3. Determine if the matrix below is nonsingular or singular. Explain your reasoning carefully and thoroughly. (15 points)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 2 & -3 \\ -1 & 1 & 1 & 0 & 0 & 2 & 3 & -3 \\ 1 & 0 & -5 & 1 & -2 & 3 & -3 & 5 \\ -1 & 0 & 5 & 0 & 1 & 0 & 2 & -4 \\ -1 & 0 & 5 & 1 & 0 & 3 & 1 & -3 \\ 1 & 0 & -5 & -1 & 3 & -6 & 5 & -6 \\ 1 & 0 & -5 & -1 & 0 & -3 & -1 & 3 \\ -2 & 1 & 6 & -1 & 0 & 1 & 2 & -2 \end{bmatrix}$$

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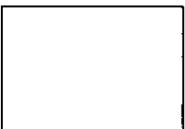
A matrix that
is not the size
8 identity matrix

By Theorem NMR1 this matrix is singular.

4. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A . It is especially important to show all of your work, so it is clear you have not used Sage. (20 points)

$$A = \begin{bmatrix} 1 & 1 & -1 & -6 & 2 \\ -1 & 0 & -2 & 4 & -2 \\ -2 & 0 & -4 & 9 & -3 \end{bmatrix} \xrightarrow{\substack{R_1 + R_2 \\ 2R_1 + R_3}} \begin{bmatrix} \boxed{1} & 1 & -1 & -6 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 2 & -6 & -3 & 1 \end{bmatrix} \xrightarrow{\substack{-R_2 + R_1 \\ -2R_2 + R_3}} \begin{bmatrix} \boxed{1} & 0 & 2 & -4 & 2 \\ 0 & \boxed{1} & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{2R_3 + R_2 \\ 4R_3 + R_1}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 6 \\ 0 & \boxed{1} & -3 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix} = B$$



5. Find all of the values of α for which the following system has a unique solution. (15 points)

$$\begin{aligned}x_1 - 3x_2 - x_3 &= 3 \\ -2x_1 + x_2 - 3x_3 &= -1 \\ x_1 + 2x_2 + \alpha x_3 &= -1\end{aligned}$$

Row-reduce, by hand, the augmented matrix of the system, carrying α as a symbol

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 3 \\ -2 & 1 & -3 & -1 \\ 1 & 2 & \alpha & -1 \end{array} \right] \xrightarrow{\substack{2R_1 + R_2 \\ -R_1 + R_3}} \left[\begin{array}{ccc|c} 1 & -3 & -1 & 3 \\ 0 & -5 & -5 & 5 \\ 0 & 5 & \alpha+1 & -4 \end{array} \right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -3 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 5 & \alpha+1 & -4 \end{array} \right]$$

$$\xrightarrow{\substack{3R_2 + R_1 \\ -5R_2 + R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & \alpha-4 & 1 \end{array} \right]$$

① $\alpha = 4 \Rightarrow$ last column will become a pivot column
 \Rightarrow by RCLS, system is inconsistent

② $\alpha \neq 4 \Rightarrow$ column 3 is a pivot column, $r=3$
 $\Rightarrow n-r = 3-3=0$ so by FVCS, unique solution

6. Suppose that: (1) A is an $n \times n$ square matrix, (2) b is a vector with n entries and (3) $\mathcal{L}S(A, b)$ has a unique solution. Prove that A is nonsingular. (15 points)

Recall Solution NM.T31

Outline: 1) $[A|b] \rightarrow [B|c]$

2) $\mathcal{L}S(A, b)$ consistent, RCLS $\Rightarrow c$ not pivot column

3) consistent, FVCS, unique solution $\Rightarrow n-r=0$
 $\Rightarrow n=r$

4) B is all pivot columns, i.e. $B = I_n$

5) $A \xrightarrow{\text{ref}} I_n$, NMRPI $\Rightarrow A$ nonsingular

