Math 290
Exam 2
Chapter V

Name: Key

Dr. Beezer
Fall 2013

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use Sage to row-reduce matrices so long as you explain your input and show your output in your work.
No other use of Sage may be used as justification for your answers.

1. Determine if the vector $\mathbf{y}$ is in the span of S, along with a complete explanation. (15 points)

$$\mathbf{y} = \begin{bmatrix} -3 \\ -8 \\ 0 \\ 6 \end{bmatrix} \qquad S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -7 \\ 1 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

From the definition of a span, want scalars $\alpha_1, \alpha_2, \alpha_3$ so that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{y}$. By SLSLC, examine system with augmented matrix and row-reduce with Sage,

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 2 & 1 & 4 & -8 \\ -1 & 2 & -7 & 0 \\ -1 & -2 & 1 & 6 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By RCLS the pivot column in column 4 implies the system is inconsistent. With no possible $\alpha_1, \alpha_2, \alpha_3$ we conclude $\mathbf{y} \notin S$.

2. Is the set of vectors R linearly independent or linearly dependent? Explain why. (10 points)

$$R = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -5 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -8 \\ -4 \\ -3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 4 \\ 3 \\ -3 \end{bmatrix} \right\}$$

LIVRN says we should put these vectors into a matrix as columns and then row-reduce (with Sage)

$$\begin{bmatrix} 1 & 1 & -8 & -2 \\ 1 & -4 & -4 & 3 \\ 1 & -5 & -3 & 4 \\ 1 & -4 & 1 & 3 \\ 0 & 3 & -2 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3 = r < n = 4$

pivot cols   # cols

So by LIVRN, the set R is linearly dependent.

3. Express solutions to the following system in "vector form" as described in Theorem VFSLS ("Vector Form of Solutions to Linear Systems"). (15 points)

Build the augmented matrix and row-reduce

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 - 5\,x_5 + 8\,x_6 &= -15 \\
x_2 + 3\,x_4 - 6\,x_5 + 2\,x_6 &= -8 \\
-x_1 + 5\,x_4 + 4\,x_5 - 3\,x_6 &= 38 \\
x_2 + x_3 + 6\,x_4 - x_5 + 6\,x_6 &= 22
\end{aligned}
\qquad
\left[\begin{array}{cccccc|c}
1 & 1 & 1 & 1 & -5 & 8 & -15 \\
0 & 1 & 0 & 3 & -6 & 2 & -8 \\
-1 & 0 & 0 & 5 & 4 & -3 & 38 \\
0 & 1 & 1 & 6 & -1 & 6 & 22
\end{array}\right]
$$

$$
\left[\begin{array}{cccccc|c}
\boxed{1} & 0 & 0 & -5 & -4 & 0 & 35 \\
0 & \boxed{1} & 0 & 3 & -6 & 0 & -6 \\
0 & 0 & \boxed{1} & 3 & 5 & 0 & 34 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & -1
\end{array}\right]
\qquad
\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 35 \\ -6 \\ 34 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 6 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$

$D = \{1,2,3,6\}$, $F = \{4,5,7\}$

4. For the set of vectors R below, find a set of vectors T meets the following two requirements and explain why your set T meets these requirements. (15 points)

(a) & (b) are the conclusion of Theorem BS, so make a matrix with vectors as columns and row-reduce

(a) $T = \langle R \rangle$

(b) T is linearly independent

$$
R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -8 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ -4 \end{bmatrix} \right\}
\text{(matrix)} \xrightarrow{\text{rref}}
\begin{bmatrix}
1 & 0 & 3 & 0 & -2 & 2 \\
0 & 1 & 4 & 0 & -3 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$D = \{1,2,4\}$ so "keep" these columns, i.e

$$
T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \\ 2 \end{bmatrix} \right\}
$$

5. Find a linear combination of vectors from S which equals $\mathbf{w}$, *using as few vectors as possible.* Include an explanation of why you are *sure* you have used the fewest possible. (15 points)

$$
\mathbf{w} = \begin{bmatrix} 5 \\ 4 \\ -4 \\ -3 \end{bmatrix} \qquad S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}
$$

we will find a linear combo via SLSLC and use <u>ideas</u> from Theorem BS's proof

$$
[\underset{\sim}{v_1}|\underset{\sim}{v_2}|\underset{\sim}{v_3}|\underset{\sim}{v_4}|\underset{\sim}{v_5}|\underset{\sim}{w}] \xrightarrow{\text{rref}}
\left[\begin{array}{ccccc|c}
1 & 0 & 4 & -3 & 0 & 2 \\
0 & 1 & -3 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\begin{aligned} D &= \{1,2,5\} \\ F &= \{3,4,6\} \end{aligned}
$$

$\underset{\sim}{v_1}, \underset{\sim}{v_2}$ & $\underset{\sim}{v_5}$ are a linearly independent set, so their span cannot be described with fewer vectors.

We can see $\underset{\sim}{w} = 2\hat{v_1} + 3\hat{v_2} + \hat{v_5}$ if we use a zero multiple of each non-pivot column for a solution.

6. Suppose that $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$. Give a careful proof that $\overline{(\alpha\mathbf{u})} = \overline{\alpha}\,\overline{\mathbf{u}}$. (Notice that this is Theorem CRSM so you are being asked to do more than just quote the statement of the theorem.) (15 points)

This is a <u>vector</u> equality, so "index" into the vectors:

For $1 \le i \le m$,

$$\begin{aligned}
[\overline{\alpha \mathbf{u}}]_i &= \overline{[\alpha \mathbf{u}]_i} && \text{conjugate of a vector} \\
&= \overline{\alpha [\mathbf{u}]_i} && \text{scalar multiplication} \\
&= \overline{\alpha}\, \overline{[\mathbf{u}]_i} && \text{property of complex numbers, multiplication \& conjugation} \\
&= \overline{\alpha}\, [\overline{\mathbf{u}}]_i && \text{conjugate of a vector} \\
&= [\overline{\alpha}\, \overline{\mathbf{u}}]_i && \text{Scalar multiplication}
\end{aligned}$$

Then by Definition CVE, $\overline{\alpha \mathbf{u}} = \overline{\alpha}\, \overline{\mathbf{u}}$

7. Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$, $S = \langle\{\mathbf{x}, \mathbf{y}\}\rangle$ and $T = \langle\{7\mathbf{x} + 3\mathbf{y},\ 2\mathbf{x} + \mathbf{y}\}\rangle$. Prove that $S = T$. (15 points)

This is a set equality, so show $T \subseteq S$ & $S \subseteq T$ (Definition SET)

<u>$T \subseteq S$</u> Grab $\mathbf{u} \in T$. Then there are scalars so that

$$\mathbf{u} = \beta_1(7\mathbf{x} + 3\mathbf{y}) + \beta_2(2\mathbf{x} + \mathbf{y}) = \cdots = (7\beta_1 + 2\beta_2)\mathbf{x} + (3\beta_1 + \beta_2)\mathbf{y}$$

The final expression implies $\mathbf{u} \in S$.

<u>$S \subseteq T$</u> Grab $\mathbf{v} \in S$. Then have scalars α_1, α_2,

we want ρ_1 & ρ_2 to do this

$$\mathbf{v} = \alpha_1 \mathbf{x} + \alpha_2 \mathbf{y} = \rho_1(7\mathbf{x} + 3\mathbf{y}) + \rho_2(2\mathbf{x} + \mathbf{y})$$
$$= (7\rho_1 + 2\rho_2)\mathbf{x} + (3\rho_1 + \rho_2)\mathbf{y}$$

Solve system:

$$\begin{aligned} 7\rho_1 + 2\rho_2 &= \alpha_1 \\ 3\rho_1 + \rho_2 &= \alpha_2 \end{aligned} \qquad \left[\begin{array}{cc|c} 7 & 2 & \alpha_1 \\ 3 & 1 & \alpha_2 \end{array}\right] \xrightarrow{-2R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & \alpha_1 - 2\alpha_2 \\ 3 & 1 & \alpha_2 \end{array}\right]$$

$$\xrightarrow[-3R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 0 & \alpha_1 - 2\alpha_2 \\ 0 & 1 & -3\alpha_1 + 7\alpha_2 \end{array}\right] \quad \text{So} \quad \begin{aligned} \rho_1 &= \alpha_1 - 2\alpha_2 \\ \rho_2 &= -3\alpha_1 + 7\alpha_2 \end{aligned}$$

implies $\mathbf{v} \in T$