Abstract Algebra Project - Modules, the Jacobson Radical, and Noncommutativity

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Abstract:

This paper is meant as an introduction into some futher topics in ring theory that we were not able to cover these past two semesters out of Judson. The main topics that will be covered are an introduction into module theory along with a proof of *Schur's Lemma*. The subsequent section deals more heavily with ideals of a ring R and attempts to introduce a new object, called the radical of a ring, where we specifically study the radical called the *Jacobson Radical*. Along with the section on modules, the Jacobson Radical helps us to try and categorize more 'special' algebraic objects. Throughout the paper, there is a flavor of noncommutativity, however, a final section at the end will give some more examples and information on some basic differences between commutative and noncommutative rings. Unfortunately, since noncommutative rings are not as well understood, it is not reasonable to discuss the strucutral difference of commutative versus noncommutative rings.

Modules:

We first begin studying the properties of a module over a ring R, or more succinctly named, an R-module. As an intuitive way of understanding what an R-module is, we can think of it as a vector space over a ring. Additionally, the definition below of a right module is the ring analogue of a group acting on a set where the ring acts on the right. More formally;

<u>Definition 1:</u> The abelian group M under addition is said to be a (right) module over a ring R, or an R-module if there is a mapping from $M \times R$ to M (sending (m, r) to mr) such that:

1.) $m(a + b) = ma + mb$

2.) $(m_1 + m_2)a = m_1a + m_2a$

3.) $(ma)b = m(ab)$

for all $m, m_1, m_2 \in M$ and all $a, b \in R$.

Note: For (left) R-modules, the map has domain $R \times M$.

If R has a unit element 1 such that $m1 = m$ for all $m \in M$, then M is said to be a *unitary* R-module.

If we let M be a (right) module over R and N a subgroup of M then we say that N is a (right) submodule of M if whenever $n \in N$ and $x \in R$, then $xn \in N$. We now define the set of endomorphisms of an additive group M. This set, which will be called $E(M)$, is all of the homomorphisms from M to M. In order to make $E(M)$ into a group we define the operation $(+)$ as follows; if we have two homomorphisms f, and g from M to M then we define $f + g : M \to M$ to be the map so that for all $x \in M$:

$$(f + g)(x) = f(x) + g(x)$$

The actual proof that $E(M)$ is indeed a group will be omitted due to its length. Additionally, if we introduce the multiplication operation as function composition, then we can subsequently convert $E(M)$ into a ring.

As an example of an R-module, let M be an abelian group under addition and let R be a subring of $E(M)$. Then M is an R-module if to each $f \in R$ and $m \in M$ we associate the elements $fm = f(m)$ and $f(m) \in M$ by f being a homomorphism.

However, if we take the converse of the above and are given a ring R and an R-module M and for every $a \in R$ we associate the mapping $\lambda_a : M \to M$ so that $\lambda_a(m) = am$ for $m \in M$, then the association $a \to \lambda_a$ is a ring homomorphism that takes R into $E(M)$. However, since the ring homomorphism $a \to \lambda_a$ may not be injective, when dealing with a module R, then we cannot in general view R as a subring of $E(M)$.

Remember that a right ideal of a ring R is a subring of R that we call I such that $Ir \subset I$ for all $r \in R$. However, as opposed to our normal view of ideals in commutative rings, we must be careful when talking about ideals since we have both left and right ideals which need not be the same.

To get an idea of what a module is, let us create one from the ring R itself:

Let R be any ring and let ρ be a right ideal of R. Then we let R/ρ be the quotient group of R by ρ which is considered an additive group since elements of R/ρ are of the form $x + \rho$ where $x \in R$. Note that since ρ is not a two-sided ideal of R, then R/ρ is not in general a ring, but it does at least carry the stucture of an R-module. In order to verify this, we define $(x+\rho)r \equiv xr+\rho$ for all $x + \rho \in R/\rho$ and for all $r \in R$. Due to ρ being a right ideal, the operation makes sense. Verification of the module properties is routine.

As for some more examples of modules, consider:

1.) If R is a ring and I is any right ideal in R, then I is a (right) R-module. Similarly, left ideals are (left) R-modules.

2.) If we take the set of all $n \times n$ matrices with real entries (which forms a ring R), then normal Euclidean space $\mathbb{R}^n$ forms a (left) R-module if the operation in our module is defined as matrix multiplication.

3.) Any additive abelian group M is a $\mathbb{Z}$-module.

Of course, we note that vector spaces over fields are examples of very nicely behaved modules. Indeed, one nice result of being a vector space over a field is that only the zero element of the field can 'annihilate' a nonzero vector. That is, if $Mr = (0)$, then $r = 0$. However, in a general module M over an arbitrary ring R, we can have that $Mr = (0)$ for some $r \neq 0$ in R. This fact motivates some terminology and the next definition. We say that M is a *faithful* R-module (or R acts *faithfully* on M) if $Mr = (0)$ implies $r = 0$. Next we set up a way to measure the 'lack of fidelity' of R on M by defining what is called the annihilator.

<u>Definition 2:</u> If M is an R-module then $A(M) = \{x \in R | mx = 0, \forall m \in M\}$. We call $A(M)$ the (right) annihilator of M in R.

The object $A(M)$ is called the (right) annihilator because of the side the ring acts on and since the elements in $A(M)$ are those elements which send every element in R which send every $m \in M$ to the zero element, thus 'annihilating' them.

Indeed, it should be relatively clear that if $A(M)$ is 'small', then R acts relatively faithfully on M. However, as the size of $A(M)$ increases, our ring acts less and less faithfully.

<u>Lemma 1:</u> $A(M)$ is a two sided ideal of R. Moreover, M is a faithful $R/A(M)$-module.

Proof. From our definition of what it means to be a (right) R-module, $A(M)$ is clearly a right ideal.

To show that $A(M)$ is also a left ideal we take $r \in R$ and $a \in A(M)$. Next, it follows that $Mr(a) = (Mr)a \subset Ma \subset (0)$, thus $ra \in A(M)$. Thus $A(M)$ is a two-sided ideal of R.

Next we attempt to make M an $R/A(M)$-module. For $m \in M$, $r + A(M) \in R/A(M)$, define the operation $m(r + A(M)) = mr$. So if $r + A(M) = r' + A(M)$ then $r - r' \in A(M)$ and thus $m(r - r') = 0$ for all $m \in M$ and thus $mr \equiv mr'$. Thus, it yields that $m(r + A(M)) = mr = mr' = m'(r + A(M))$ and so the operation of $R/A(M)$ on M is well-defined. Verification that our action defines the structure of an $R/A(M)$-module on M can be done by hand.

In order to see that M is a faithful $R/A(M)$-module note that if $m(r + A(M)) = 0$ for all $m \in M$ then by definition $mr = 0$ hence $r \in A(M)$. The above states that only the zero element of $R/A(M)$ annihilates all of M. □

<u>Lemma 2:</u> $R/A(M)$ is isomorphic to a subring of $E(M)$ where $E(M)$ is the set of all endomor-

phisms of the additive group of M.

Proof. If we let M be a unitary R-module and for $a \in R$ define $T_a : M \to M$ by $mT_a = ma$ for all $m \in M$. By the domain and codomain, T_a is a function and although the way that it is defined is odd, we can think of T_a as taking inputs on the left. Additionally, since M is an R-module, then T_a is an endomorphism of the additive group of M. That it, $(m_1 + m_2)T_a = m_1Ta + m_2T_a$ for all $m_1, m_2 \in M$.

Now if we consider the mapping $\phi : R \to E(M)$ defined by $\phi(a) = T_a$ for all $a \in R$, then if we go back to the definition of an R-module, we realize that $\phi(a+b) = T_{a+b} = 1T_{a+b} = a+b = 1a + 1b = T_a + T_b = \phi(a) + \phi(b)$ and $\phi(ab) = T_{ab} = 1T_{ab} = ab = 1a1b = T_aT_b = \phi(a)\phi(b)$. Thus, ϕ is a ring homomorphism of R into $E(M)$. As with any homomorphism, it is useful to consider the kernel, $\ker(\phi)$. Indeed, in this case, it is possible to give a relatively satisfactory classification of the kernel. To begin with, if $a \in A(M)$ then $Ma = (0)$ by definition so $0 = T_a = \phi(a)$. Thus $a \in \ker(\phi)$. However, if we take $a \in \ker(\phi)$ then $T_a = 0$ which leads to $Ma = MT_a = (0)$, and hence $a \in A(M)$. We have now shown that the image of R in $E(M)$ is isomorphic to $R/A(M)$. □

It is interesting to note that if M is a faithful R-module and thus $A(M) = (0)$, Lemma 2 states that we can consider R as a subring of the ring of endomorphisms of M as an additive group. This is because when $A(M) = (0)$, we can think of $R/A(M)$ as just R.

From the relation of the R-module M with the ring R, we have come up with these elements, T_a as a ranges over R, where $T_a \in E(M)$. However, we do not know how these elements are arranged in $E(M)$ and whether or not there any special structure that we can create that might be useful? Indeed, we can define an object that contains all the endomorphisms that commute with all the T_a's.

<u>Definition 3:</u> The commuting ring of R on M is

$$C(M) = \{\theta \in E(M) | T_a\theta = \theta T_a \text{ for all } a \in R\}$$

$C(M)$ is most certainly a subring of $E(M)$. If $\theta \in E(M)$ then for any $m \in M$ and $a \in R$,

$$(m\theta)a = (m\theta)1a = (m\theta)T_a = m(\theta T_a) = m(T_a\theta) = (m1T_a)\theta = (ma)\theta$$

That is, θ is not only an endomorphism of M as an additive group from definition, but is in fact a homomorphism of M into itself as an R-module. Thus $C(M)$ is the ring of all *module* endomorphisms of M.

<u>Definition 4:</u> M is said to be an irreducible (or simple) R-module if $MR \neq (0)$ and if the

only submodules of M are (0) and M.

Note: If M is a simple R-module, then M is cyclic. To show this, take a nonzero element $m \in M$. Let $N = \langle m \rangle$ be the cyclic subgroup of M generated by the element m. Then since $N \neq \langle 0 \rangle$ and M is simple, $M = N$.

For an irreducible R-module M, the commuting ring happens to be rather special. The following Theorem is from the early 1900's and is known as *Shur's Lemma*.

Theorem 1: If M is an irreducible R-module then $C(M)$ is a division ring.

Proof. In order to prove this theorem, all we need to show is that for every nonzero element in $C(M)$, there exists an inverse in $C(M)$. However, by the elements in $C(M)$, we must show that for $\theta \neq 0 \in C(M)$ then θ is invertible in $E(M)$. This is due to the fact that if $\theta^{-1} \in E(M)$ then since $\theta T_a = T_a\theta$ by the commuting ring, we obtain $T_a\theta^{-1} = \theta^{-1}T_a$ which results in $\theta^{-1} \in C(M)$.

Now we must show that $\theta^{-1} \in E(M)$. Suppose that $\theta \neq 0 \in C(M)$, so if $W = M\theta$ then for all $r \in R$, we get that $Wr = WT_r = (M\theta)T_r = (MT_r)\theta \subset M\theta = W$. Thus W is a submodule of M. Since $\theta \neq 0$ by the irreducibility of M then we can conclude that $W\theta = M$ or, equivalently, that θ is surjective. To claim that θ is injective, we note that $\ker(\theta)$ is a submodule of M and is not all of M since $\theta \neq 0$. Therefore, by M's irreduciblity, $\ker(\theta) = (0)$. By the surjective and injectiveness of θ, we conclude that $\theta^{-1} \in E(M)$. Thus Schur's Lemma has been proven. □

As a final lemma for this section, we attempt to give a characterization of all of the possible irreducible modules of a ring R. However, due to the length of its proof, the lemma shall just be stated as is.

Lemma 3: If M is an irreducible R-module then M is isomorphic as a module to R/ρ for some maximal right ideal ρ of R. Moreover there is an $a \in R$ such that $x - ax \in \rho$ for all $x \in R$. Conversely, for every maximal right ideal ρ of R, R/ρ is an irreducible R-module.

Jacobson Radical:

In trying to figure out the structure theory for a category of algebraic objects it is useful to see when classes of items are 'nice' and to be able to meaure the amount of 'niceness'. After identifying these nicely behaved objects, we would like a tool to go from a general object to a better behaved object. In this vein of thought, we introduce what is called the radical of a ring. The radical of a ring can be thought of as all of the ideal consisting of poorly behaved elements of the ring and although there have been many examples of radicals starting in the early 1900's, we will focus on the Jacobson radical, which is one of the more important examples.

Definition 5: The radical of R, written as $J(R)$, is the set of all elements of R which annihilate all the irreducible R-modules. If R has no irreducible modules we put $J(R) = R$. This particular definition of the radical of a ring is called the *Jacobson Radical* since there are other types of radicals of a ring.

Take note that $J(R) = \bigcap_M A(M)$ where M is the set of al irreducible R-modules. Additionally, from Lemma 1, the $A(M)$'s are two-sided ideals of R and thus $J(R)$ is also a two-sided ideal of R. We could define a left radical and a right radical of R from using a left or right annihilator in our definition, but it turns out that they will both be the same and thus no distinction is made.

Definition 6: A right ideal ρ of R is said to be regular if there is an $a \in R$ such that $x - ax \in \rho$ for all $x \in R$.

This definition is motivated from part of Lemma 3, namely the existence of some element $a \in R$ so that $x - ax \in \rho$ for all $x \in R$ where ρ is an ideal.

Definition 7: If ρ is a right ideal of R then $(\rho : R) = \{x \in R | rx \in \rho \; \forall r \in R\}$.

If we let ρ be a maximal regular right ideal in R and let $M = R/\rho$, then what is the annihilator of M, $A(M)$? Let us pull an element x out of $A(M)$, which satisfies $Mx = (0)$ and thus $(r + \rho)x = \rho$ for all $r \in R$. However, this is equivalent to saying that $Rx \subset \rho$ and thus $A(M) \subset (\rho : R)$. Due to ρ being regular, there exists an $a \in R$ so $x - ax \in \rho$ for all $x \in R$. Taking a specific x, one such that $x \in (\rho : R)$, since $ax \in Rx \subset \rho$, then $x \in \rho$. Therefore, since we have done both subsetting relations, $A(M) = (\rho : R)$ is the largest two-sided ideal of R in ρ.

Looking back at Lemma 3, it is now clearer as to the motivation behind the following theorem.

Theorem 2: $J(R) = \cap(\rho : R)$ where ρ runs over all the regular maximal right ideals of R, and where $(\rho : R)$ is the largest two-sided ideal of R lying in ρ.

In the long run, it would be nice to make the conditions of Theorem 2 more stringent. In fact, this section will conclude with the proof of the more stringent theorem which relates the Jacobson Radical to the intersection of the regular maximal right ideals of a ring. Finally, a few new definitions are introduced to provide a wider array of algebraic objects to consider.

Lemma 4: If ρ is a regular right ideal of R then ρ can be embedded in a maximal right ideal of R which is regular.

Proof. Let $a \in R$ so that $x - ax \in \rho$ for all $x \in R$. We know that such an a exists by the regularity of ρ. We can eliminate the possibility of $a \in \rho$ since if $a \in \rho$, then $ax \in \rho$ and $x \in \rho$ for all $x \in R$ which leads to $\rho = R$.

Let $\mathcal{M}$ be the set of all proper right ideals of R which contain ρ. If $\rho' \in \mathcal{M}$, then $a \notin \rho'$ since otherwise we would get that $x - ax \in \rho \subset \rho'$ and thus $\rho' = R$. Through the application of Zorn's Lemma, we can get a ρ_0 that is maximal in $\mathcal{M}$. Note that ρ_0 is regular since $x - ax \in \rho \subset \rho_0$. Additionally, from our use of Zorn's Lemma, ρ_0 is a maximal right ideal of R. □

Theorem 3: $J(R) = \cap\rho$ where ρ runs over all the maximal regular right ideals of R.

Proof. By Theorem 2, $J(R) = \cap(\rho : R)$ and because $(\rho : R) \subset \rho$ then we get that $J(R) \subset \cap\rho$ where ρ varies over all of the regular maximal right ideals of R.

For the other direction, let $\tau = \cap\rho$ and let $x \in \tau$. Our claim is that the set $\{xy + y | y \in R\}$ is all of R. If this is not the case, since our set has the same structure as a regular right ideal (with $a = -x$), then from Lemma 4, this ideal would be contained in a regular maximal right ideal ρ_0. Due to the fact that $x \in \cap\rho$, we know that $xy \in \rho_0$ and thus $y \in \rho_0$ for all $y \in R$, which is a contradiction. Thus, $\{xy + y | y \in R\} = R$ and in particular, for some $w \in R$, $-x = w + xw$, or equvalently, $x + w + xw = 0$. Now if $\tau \not\subseteq J(R)$ then for some irreducible R-module M it must be the case that $M\tau \neq (0)$, and thus $m\tau \neq 0$ for some $m \in M$. As a nonzero submodule of M, we obtain that $m\tau = M$. Therefore, for some $t \in \tau$, $mt = -m$ due to the fact that $t \in \tau$ and we have seen above that there is an $s \in R$ so that $t + s + ts = 0$.

We start with $0 = m(s + t + ts) = ms + mt + mts = ms - m - ms = -m$ and obtain the contradiction that $m = 0$. Therefore, $M\tau = (0)$ for all of the irreducible R-modules M thereby placing them in $J(R)$. □

Definition 8: An element $a \in R$ is said to be right-quasi-regular if there is an $a' \in R$ such that $a + a' + aa' = 0$. We call a' a right-quasi-inverse of a.

Definition 9: R is said to be semi-simple if $J(R) = (0)$.

Noncommutative Rings

Since noncommutativity is quite evident throughout this paper, more general examples will be provided of noncommutative rings rather than a discussion of the difference of noncommutative vs commutative rings. Of course, the difference between these two types of rings starts at a very basic level and although we have often relied on commutativity in order to force powerful results in group and ring theory, there is a structural framework of noncommutative rings. However, even though there is a structure in place, the serious study of noncommutative rings only began very recently, in the early to mid 1900's by people like Brauer, Jacobson, Herstein, and Cohn. Due to the relatively recent study of noncommutative rings along with their relative nastiness as compared to commutative rings, these rings are less well understood than their commutative counterparts.

These noncommutative rings have some interesting features about them that are not immediately evident. For example, in a noncommutative ring R that is simple, R can have non-trivial proper right ideals along with non-trivial left ideals and yet lack a non-trivial proper two-sided ideal of R by definition of simplicity. Although these rings can appear to be very abstract, the ring of $n \times n$ matrices over a field is often used in physics and is itself an example of a noncommutative ring.

Examples:

1.) The set of 2×2 matrices over a field is a noncommutative ring that is simple.

2.) The Quaternions, $\{1, -1, i, -i, j, -j, k, -k\}$.

3.) Similar to 1.), if we take the set of 2×2 matrices with entries in $\mathbb{Z}_2$, then we obtain a noncommutative ring, this time one that is finite with 16 elements.

Conclusions: Modules are a weaker form of vector spaces over rings and while there are similarities, the differences and the ability to have a left or right module give us enough information to create new objects (one of them being the annihilator amongst others) which is useful in the study of noncommutative rings. The annihilator of a ring is exactly the tool employed in a discussion of the Jacobsn radical, where this radical can be thought of as all of the bad elements of a ring (the ones that annihilate irreducible R-modules). A few results about the Jacobson radical and tying it together with regular maximal right ideals of R are shown to tie together the ideas of modules, noncommutativity and the Jacobson radical. Finally, a few more examples and some background information on noncommutative rings were included in order to provide a decent base of examples of these oddly behaved rings.

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