Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use reduced row-echelon from Sage as justification for parts of your answers, so long as you explain your input and list the output in your solution.

1. Solve the following system of equations *using an inverse of a matrix*. No credit will be given for solutions obtained using a different method. (15 points)

$$\begin{aligned} x_1 - 4x_2 + 3x_3 &= -5 \\ -x_1 + 5x_2 - 5x_3 &= 3 \\ -x_1 + 4x_2 - 2x_3 &= 9 \end{aligned}$$

SLEMM-ified $A\underset{\sim}{x} = \underset{\sim}{b}$ with $A = \begin{bmatrix} 1 & -4 & 3 \\ -1 & 5 & -5 \\ -1 & 4 & -2 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -5 \\ 3 \\ 9 \end{bmatrix}$

$[A|I_3] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 4 & 5 \\ 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$ so A is nonsingular (Theorem NMRRI) and by Theorem CINM we have A^{-1}.

Theorem SNCM gives the unique solution as

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b} = \begin{bmatrix} 10 & 4 & 5 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$$

2. Compute the inverse of the matrix A. (10 points)

$$A = \begin{bmatrix} -1 & 0 & 6 & -4 \\ -1 & -2 & 1 & 1 \\ -2 & -3 & 5 & -1 \\ 2 & 2 & -3 & -1 \end{bmatrix}$$

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem NMRRI, A is singular.

By Theorem NI, A is not invertible

3. Compute the requested versions of row and column spaces for the matrix B. (50 points)

$$B = \begin{bmatrix} 2 & -5 & 9 & 33 & 1 & -8 \\ -3 & 24 & -63 & -132 & -11 & 38 \\ 1 & -7 & 18 & 39 & 3 & -11 \\ 0 & -7 & 21 & 35 & 4 & -11 \\ 0 & -2 & 6 & 10 & 1 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 & 4 & 0 & -1 \\ 0 & 1 & -3 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The column space of B, using only the definition. Span of the 6 columns.

$$C(B) = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 24 \\ -7 \\ -7 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -63 \\ 18 \\ 21 \\ 6 \end{bmatrix}, \begin{bmatrix} 35 \\ -132 \\ 35 \\ 35 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ -11 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 38 \\ -11 \\ -11 \\ -3 \end{bmatrix} \right\} \right\rangle$$

(b) The column space of B, as the span of a linearly independent set containing only columns of B.

Pivot columns $= D = \{1, 2, 5\}$ By Theorem BCS, use these columns

$$C(B) = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 24 \\ -7 \\ -7 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -11 \\ 3 \\ 4 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(c) The column space of B, as the span of a linearly independent set obtained from the row space of a different matrix.

$C(B) = R(B^t)$ then use BRS

$$B^t \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \\ 5/3 \end{bmatrix} \right\} \right\rangle$$

(d) The column space of B, as the span of a linearly independent set derived from the extended echelon form of B.

$$[B \mid I_5] \xrightarrow{\text{rref}} \left[\begin{array}{cccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 1 & -2 \end{array}\right] \leftarrow L$$

Theorem FS

$$C(B) = N(L) = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(e) The row space of B, as the span of a linearly independent set.

Apply Theorem BRS, using rref at very top. "Keep" non zero rows.

$$R(B) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \right\rangle$$

4. Suppose that A and B are $m \times n$ matrices. Prove that $A + B = B + A$, giving a reason for each step of your proof. (10 points)

For $1 \leq i \leq m$, $1 \leq j \leq n$

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad \text{Definition MA}$$

$$= [B]_{ij} + [A]_{ij} \qquad \text{Property CACN}$$

$$= [B+A]_{ij} \qquad \text{Definition MA}$$

So by Definition ME, $A+B = B+A$.

5. Suppose A is an $m \times n$ matrix and B, C are $n \times p$ matrices. Prove that $A(B + C) = AB + AC$. (This is Theorem MMDAA, so do more than just quote this result.) (15 points) For $1 \leq i \leq m$, $1 \leq j \leq p$

$$[A(B+C)]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B+C]_{kj}$$

$$= \sum_{k=1}^{n} [A]_{ik}([B]_{kj} + [C]_{kj})$$

$$= \sum_{k=1}^{n} [A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj}$$

$$= \sum_{k=1}^{n} [A]_{ik}[B]_{kj} + \sum_{k=1}^{n} [A]_{ik}[C]_{kj}$$

$$= [AB]_{ij} + [AC]_{ij}$$

$$= [AB+AC]_{ij}$$

So by Defn ME, $A(B+C) = AB + AC$.