Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2014
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may compute reduced row-echelon form and matrix inverses with Sage, and provide this as justification for parts of your answers, so long as you explain your input and list the output in your solution.

1. The set $B = \{2x^2 + 2x + 1,\ 4x^2 + 5x + 2,\ -3x^2 - 4x - 2\}$ is a basis of the vector space, P_2, of polynomials with degree at most 2 (you may assume this). Write the vector $\mathbf{u} = x^2 + x + 1$ as a linear combination of the vectors in B. (15 points)

We desire: $x^2+x+1 = \alpha_1(2x^2+2x+1) + \alpha_2(4x^2+5x+2) + \alpha_3(-3x^2-4x-2)$

$= (2\alpha_1+4\alpha_2-3\alpha_3)x^2 + (2\alpha_1+5\alpha_2-4\alpha_3)x + (\alpha_1+2\alpha_2-2\alpha_3)$

Equating coefficients

$2\alpha_1+4\alpha_2-3\alpha_3=1$
$2\alpha_1+5\alpha_2-4\alpha_3=1$
$\alpha_1+2\alpha_2-2\alpha_3=1$

augmented matrix $\xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

So $\alpha_1=1$, $\alpha_2=-1$, $\alpha_3=-1$. This answer is unique since B is a basis.

2. $S = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| 3a + 2b = 0 \right\}$ is a subset of the vector space $\mathbb{C}^2$. Prove that S is a subspace of $\mathbb{C}^2$. (15 points)

1) $S \neq \emptyset$. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$ since $3(0)+2(0)=0$.

2) Additive Closure. Suppose $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are in S, so we know $3x_1+2x_2=0$, $3y_1+2y_2=0$. Also $\underset{\sim}{x}+\underset{\sim}{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$

Then $3(x_1+y_1)+2(x_2+y_2) = 3x_1+3y_1+2x_2+2y_2 = (3x_1+2x_2)+(3y_1+2y_2) = 0+0 = 0$

So $\underset{\sim}{x}+\underset{\sim}{y} \in S$

3) Scalar Closure. Suppose $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S$, so $3x_1+2x_2=0$.
Know $\alpha\underset{\sim}{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$. Then

$3(\alpha x_1)+2(\alpha x_2) = \alpha(3x_1)+\alpha(2x_2) = \alpha(3x_1+2x_2) = \alpha\cdot 0 = 0$

So $\alpha\underset{\sim}{x} \in S$.

3. The set $U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \; a - 3b + 2c = 0 \right\}$ is a subspace of $\mathbb{C}^3$ (you may assume this). (40 points)

(a) Find a minimal spanning set for U.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3b-2c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \text{So } U = \left\langle \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(b) Prove that your set from part (a) is linearly independent.

$$\alpha_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 3\alpha_1 - 2\alpha_2 = 0 \\ \alpha_1 = 0 \\ \alpha_2 = 0 \end{matrix} \Rightarrow \alpha_1 = \alpha_2 = 0$$

So the only RLD is a trivial one.

(c) What is the dimension of U?

The set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis, and has size 2.

So $\dim(U) = 2$.

(d) Is $R = \left\{ \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} \right\} \subseteq U$ a basis of U? Explain. By theorem G it has the right size.

$$\alpha_1 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[\text{augmented matrix of a system}]{\text{rref of}} \alpha_1 = \alpha_2 = 0$$

Since R is linearly independent, it also spans & so is a basis

(e) Is $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \right\} \subseteq U$ a basis of U? Explain.

By theorem G, $|T| = 3 > 2 = \dim(U)$, so

T is linearly dependent and thus not a basis.

4. Suppose that V is a vector space and $\mathbf{v} \in V$. Prove that $-\mathbf{v} = (-1)\mathbf{v}$. (15 points)

Is $(-1)\mathbf{v}$ the additive inverse of $\mathbf{v}$? Let's check.

$$
\begin{aligned}
(-1)\mathbf{v} + \mathbf{v} &= (-1)\mathbf{v} + 1\mathbf{v} && \text{Property O} \\
&= [(-1)+1]\,\mathbf{v} && \text{Property DSA} \\
&= 0\mathbf{v} && \text{Grade school} \\
&= \mathbf{0} && \text{Theorem ZSSM}
\end{aligned}
$$

5. Suppose that V is a vector space and $\mathbf{v}_1, \mathbf{v}_2 \in V$. Prove that $X = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for V if and only if $Y = \{2\mathbf{v}_1 + 5\mathbf{v}_2, \mathbf{v}_1 + 3\mathbf{v}_2\}$ is a spanning set for V. (15 points)

($\Leftarrow$) Assume Y spans V. Grab $\mathbf{w} \in V$, totally generic. Then

$$
\begin{aligned}
\mathbf{w} &= \alpha_1(2\mathbf{v}_1 + 5\mathbf{v}_2) + \alpha_2(\mathbf{v}_1 + 3\mathbf{v}_2) = 2\alpha_1\mathbf{v}_1 + 5\alpha_1\mathbf{v}_2 + \alpha_2\mathbf{v}_1 + 3\alpha_2\mathbf{v}_2 \\
&= \underbrace{(2\alpha_1 + \alpha_2)}_{\text{scalars}}\mathbf{v}_1 + \underbrace{(5\alpha_1 + 3\alpha_2)}_{\text{scalars}}\mathbf{v}_2
\end{aligned}
$$

So, any $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ & $\mathbf{v}_2$, so X spans V.

($\Rightarrow$) Assume X spans V. Grab $\mathbf{w} \in V$, totally generic. Then

$$
\begin{aligned}
\mathbf{w} &= \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 = \beta_1\big(3(2\mathbf{v}_1 + 5\mathbf{v}_2) - 5(\mathbf{v}_1 + 3\mathbf{v}_2)\big) \\
&\qquad + \beta_2\big((-1)(2\mathbf{v}_1 + 5\mathbf{v}_2) + 2(\mathbf{v}_1 + 3\mathbf{v}_2)\big) \\
&= 3\beta_1(2\mathbf{v}_1 + 5\mathbf{v}_2) - 5\beta_1(\mathbf{v}_1 + 3\mathbf{v}_2) + (-\beta_2)(2\mathbf{v}_1 + 5\mathbf{v}_2) + 2\beta_2(\mathbf{v}_1 + 3\mathbf{v}_2) \\
&= \underbrace{(3\beta_1 - \beta_2)}_{\text{scalars}}(2\mathbf{v}_1 + 5\mathbf{v}_2) + \underbrace{(-5\beta_1 + 2\beta_2)}_{\text{scalars}}(\mathbf{v}_1 + 3\mathbf{v}_2)
\end{aligned}
$$

Can "discover" these linear combinations with two linear systems

So any $\mathbf{w}$ is a linear combination of $2\mathbf{v}_1 + 5\mathbf{v}_2$ & $\mathbf{v}_1 + 3\mathbf{v}_2$, so Y spans V.