Math 290
Exam 5
Chapters D&E

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
You may use computations from Sage as justification for your answers **only** as indicated in each question.

1. Compute the determinant of A (without using Sage). (15 points)

$$A = \begin{bmatrix} 2 & -1 & 4 & -1 \\ -2 & 2 & 0 & 2 \\ 4 & 3 & 0 & 3 \\ 0 & -1 & -3 & -1 \end{bmatrix}$$

(1) A zero in row 4, column 1

(2) Two zeros in column 3

(3) Columns 2 & 4 are equal $\xRightarrow{\text{theorem DERC}}$ $\det(A)=0$

2. Given the matrix B below, find an invertible matrix S such that $S^{-1}BS$ is a diagonal matrix. You may use Sage to compute eigenvalues and to row-reduce matrices. (15 points)

$$B = \begin{bmatrix} -4 & 12 & -18 \\ -6 & 23 & -36 \\ -3 & 12 & -19 \end{bmatrix}$$

B.eigenvalues() returns $[2, -1, -1]$

$(B-2I_3)$.rref() $\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of eigenspace for $\lambda=2$

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$$

$(B-(-1)I_3)$.rref() $\rightarrow \begin{bmatrix} 1 & -4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of eigenspace for $\lambda=-1$

$$\left\{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}\right\}$$

So $S = \begin{bmatrix} 1 & 4 & -6 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ will diagonalize B. (theorem DC)

(You could check, but this was not part of the question)

3. Consider the square matrix A. (40 points)

$$A = \begin{bmatrix} -27 & 48 & 31 & -20 \\ -25 & 47 & 25 & -13 \\ 40 & -80 & -36 & 13 \\ 42 & -84 & -42 & 18 \end{bmatrix}$$

(a) Use Sage to compute a factored version of the characteristic polynomial.

$A.\mathrm{fcp}() \Rightarrow (x-4)^2(x+3)$ (or .charpoly(), then .factor())

[Eigenvalues: $\alpha_A(4)=2$, $\alpha_A(-3)=1$]

(b) Without using Sage, determine the eigenvalues of A and their algebraic multiplicities.

Roots of characteristic polynomial with multiplicities

(c) Using only Sage's reduced row-echelon form method for computational assistance, determine all the eigenspaces of A along with their ~~algebraic~~ geometric multiplicities.

$(A-4I_4).\mathrm{rref}()$ $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{E}_A(4) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle$

$\gamma_A(4) = 1 \nearrow$ dimension

$(A+3I_4).\mathrm{rref}()$ $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{E}_A(-3) = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \right\rangle$

$\gamma_A(-3) = 1 \nearrow$ dimension

(d) Is A diagonalizable? Explain fully.

$\alpha_A(4) = 2 \neq 1 = \gamma_A(4)$

$\alpha_A(-3) = 2 \neq 1 = \gamma_A(-3)$

just one of these, with Theorem DMFE tells us A is not diagonalizable.

4. Suppose that A is an $n \times n$ matrix and $\lambda \in \mathbb{C}$. Define

$$U = \{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

Prove additive closure for U, which is one of the three conditions of checking that a set is a subspace. (15 points)

Suppose $x \in U$, $y \in U$. Then know $Ax = \lambda x$, $Ay = \lambda y$.

Consider $x+y$. Examine

$$A(x+y) = Ax + Ay = \lambda x + \lambda y = \lambda(x+y)$$

So $x+y \in U$, as desired.

5. Suppose that A, B and C are three $n \times n$ matrices that are equal to each other, except for the entries in column k. In A column k is the vector $\mathbf{x}$, in B column k is the vector $\mathbf{y}$, and in C column k is the vector $\mathbf{x}+\mathbf{y}$. Prove that $\det(C) = \det(A) + \det(B)$. (15 points)

Pictorially: $A = [\ldots|\mathbf{x}|\ldots]$ $\quad B = [\ldots|\mathbf{y}|\ldots]$ $\quad C = [\ldots|\mathbf{x}+\mathbf{y}|\ldots]$

$$\det(C) = \sum_{i=1}^{n} (-1)^{i+k} [x+y]_i \det(C(i|k))$$

$$= \sum_{i=1}^{n} (-1)^{i+k} \left([x]_i + [y]_k\right) \det(C(i|k))$$

$$= \sum_{i=1}^{n} (-1)^{i+k} [x]_i \det(C(i|k)) + \sum_{i=1}^{n} (-1)^{i+k} [y]_i \det(C(i|k))$$

$$= \sum_{i=1}^{n} (-1)^{i+k} [x]_i \det(A(i|k)) + \sum_{i=1}^{n} (-1)^{i+k} [y]_i \det(B(i|k))$$

$$= \det(A) + \det(B)$$