Application of Linear Algebra

Fuzzy Leontief Input Output Models

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Basics of Leontief Input Output Model

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- Used to answer: What level of total output does an economy need to produce to meet a given final demand using a given technology?

Notation for LIO Model

a_{ij} is the amount of good i used in the production of one unit of good j where $i = 0, 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, a_{0j} represents the labor allocated to the production of the jth good's output.
$A = [a_{ij}]$, is the input coefficient matrix or the technological matrix.
C, the final consumption vector whose entries, c_i are the final consumption of the ith good
X whose entries denote the gross production of the ith good.
x_i is the total output of the ith good, and let x_0 be the total "production" of a primary non-produced good.
Note: For our purposes we neglect the primary input, a_{0j}, c_0, x_0, for most of this presentation though its existence is important later.

Crisp LIO Model

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and profitability

$$\tilde{\mathbf{p}} > A\tilde{\mathbf{p}}. \tag{5}$$

Theorem

For any $n \times n$ matrix B with $b_{ij} \leq 0$ for all $i \neq j$, the following three conditions are equivalent:

(a) *there exists an $\mathbf{x} \in \mathbb{R}^n_+$, such that $B\mathbf{x} > 0$*

(b) *B is a P matrix, that is B has strictly positive principal minors. A principal minor is the determinate of a submatrix formed from by "deleting" the same rows and columns form B. This is called a principal submatrix.*

(c) *$B^{-1} \geq 0$, that is $b_{ij} \geq 0$ for all i and j.*

Fuzzy Numbers

Definition

A fuzzy number, $\bar{A}$, in $\mathbb{R}$, with membership function $\mu_{\bar{A}}(x)$, is a fuzzy subset of $\mathbb{R}$ that is both convex and normal.

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A trapezoidal fuzzy number can be completely described by a quadruplet $\bar{A} = (a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$, and whose membership function is defined to be,

$$\mu_{\bar{A}}(x) \begin{cases} = \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ = 1, & a_2 \leq x \leq a_3 \\ = \frac{a_3-x}{a_3-a_2}, & a_3 \leq x \leq a_4 \\ = 0, & x \geq a_3. \end{cases}$$

Fuzzy Numbers

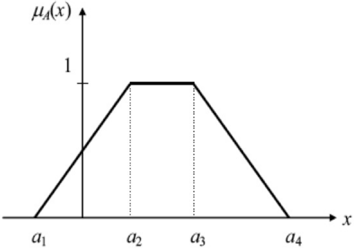

Figure : A trapezoidal fuzzy number $\bar{A} = (a_1, a_2, a_3, a_4)$

Fuzzy Numbers

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α-cuts:

$$\begin{aligned} A_\alpha &= [a_1^{(\alpha)}, a_4^{(\alpha)}] \\ &= [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]. \end{aligned}$$

Fuzzy Leontief Input Output Model

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Fuzzy numbers allow us to capture that uncertainty.
How? Because of close relationship between the level of presumption and an interval of confidence.

Fuzzy Leontief Input Output Model

- $\bar{A} = [\bar{a}_{ij}]$ is a $n \times n$ matrix of fuzzy numbers $\bar{a}_{ij} = (a_{ij1}|a_{ij2}, a_{ij3}|a_{ij4})$ where $0 \leq a_{ij1} \leq a_{ij2} \leq a_{ij3} \leq a_{ij4} \leq 1$ represent the fuzzy input coefficients.
- $\bar{\mathbf{C}} = [\bar{c}_i]$ be an $n \times 1$ vector where $\bar{c}_i = (c_{i1}|c_{i2}, c_{i3}|c_{i4})$ and $\bar{c}_i$ is non-negative. $\bar{\mathbf{C}}$ is a fuzzy vector of final consumption demand.
- $\bar{\mathbf{X}} = [\bar{x}_i]$ be an $n \times 1$ where $\bar{x}_i = (x_{i1}|x_{i2}, x_{i3}|x_{i4})$ and $\bar{x}_i$ is non-negative. $\bar{\mathbf{X}}$ is a fuzzy vector of total output for industries in this economy [1,2,3].

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Just as in the crisp case we are seeking to find a level of total output that will satisfy both final and intermediate demand. We are finding $\bar{\mathbf{X}}$

$$\bar{\mathbf{X}} = (I - \bar{A})^{-1}\bar{\mathbf{C}}. \tag{6}$$

Fuzzy Leontief Input Output Model

Fuzzy arithmetic is more easily preformed in terms of intervals of confidence for the level of presumption $\alpha \in [0, 1]$, called α-cuts. We define,

$$\bar{a}_{ij}^{\alpha} = [a_{ijl}^{\alpha}, a_{iju}^{\alpha}]$$
$$\bar{c}_{i}^{\alpha} = [c_{il}^{\alpha}, c_{iu}^{\alpha}]$$
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Fuzzy arithmetic based on α-cut arithmetic becomes interval arithmetic and we are now finding $\mathbf{X}_l^{\alpha}$ and $\mathbf{X}_u^{\alpha}$ such that;

$$\mathbf{X}_l^{\alpha} = (I - A_l^{\alpha})^{-1}\mathbf{C}_l^{\alpha}, \quad (7)$$
$$\mathbf{X}_u^{\alpha} = (I - A_u^{\alpha})^{-1}\mathbf{C}_u^{\alpha}. \quad (8)$$

[1]

Fuzzy Leontief Input Output Model

We are guaranteed that a fuzzy economy exists by the following theorem.

Theorem

If $\sum_{i=1}^{n} a_{ij4} < 0$ for all j, then the fuzzy input output model exists for this economy. [1]

A Numerical Example

A Simple Two Sector Economy

	Industries			
	Agriculture	Manufacturing	Final Consumption ($\bar{\mathbf{C}}$)	Gross Output ($\bar{\mathbf{X}}$)
Agriculture	$(0.25/0.3/0.35)$	$(0.3/0.4/0.5)$	$(60/65, 75/80)$	$\bar{x}_1$
Manufacturing	$(0.4/0.45, 0.55/0.60)$	$(0.2/0.25, 0.35/0.4)$	$(50/55, 65/70)$	$\bar{x}_2$
Outside Inputs	$(0.1/0.2/0.3)$	$(0.2/0.3/0.4)$		
Total	$(0.75/0.95, 1.05/1.25)$	$(0.7/0.95, 1.05/1.3)$		

A Numerical Example

Does Theorem 2 hold:

$a_{114} + a_{214} < 1$ and $a_{214} + a_{224} < 1$ condition is met. Therefore guaranteed a solution to $\bar{x}_1$ and $\bar{x}_2$.

Assume the membership function for $\bar{a}_{ij}$ is linear.

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Assume the membership function for $\bar{a}_{ij}$ is linear.

Compute the alpha-cuts for the fuzzy input coefficient matrix and fuzzy final consumption vector.

$$\bar{\mathbf{X}}_l^\alpha = \begin{bmatrix} x_{1l}^\alpha \\ x_{2l}^\alpha \end{bmatrix}, \bar{A}_l^\alpha = \begin{bmatrix} 0.25 + 0.05\alpha & 0.4 + 0.1\alpha \\ 0.4 + 0.05\alpha & 0.2 + 0.05\alpha \end{bmatrix}, \bar{\mathbf{C}}_l^\alpha = \begin{bmatrix} 60 + 5\alpha \\ 50 + 5\alpha \end{bmatrix}$$

$$\bar{\mathbf{X}}_u^\alpha = \begin{bmatrix} x_{1u}^\alpha \\ x_{2u}^\alpha \end{bmatrix}, \bar{A}_u^\alpha = \begin{bmatrix} 0.35 - 0.05\alpha & 0.5 - 0.1\alpha \\ 0.6 - 0.05\alpha & 0.4 - 0.05\alpha \end{bmatrix}, \bar{\mathbf{C}}_u^\alpha = \begin{bmatrix} 80 - 5\alpha \\ 70 - 5\alpha \end{bmatrix}$$

Numerical Example

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Example:

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Example: var('a')

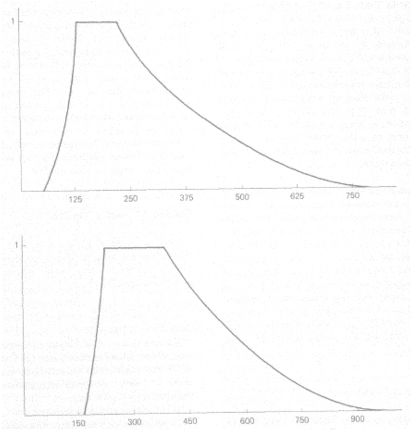
Au = matrix(SR, [[0.35 -0.05*a , 0.5 - 0.1*a],
 [0.6 - 0.05*a , 0.4 - 0.05*a]])

Cu = matrix(SR, [[80 + 5*a],
 [70 + 5*a]])

$$\bar{\mathbf{X}}_l^{\alpha} = \begin{bmatrix} \frac{-100(a^2+32a+272)}{(a^2+55a-176)} \\ \frac{-100(25a+246)}{(a2+55a-176)} \end{bmatrix}$$

$$\bar{\mathbf{X}}_u^{\alpha} = \begin{bmatrix} \frac{100(a^2-10a-332)}{(a^2-59a-36)} \\ \frac{-100(23a+374)}{(a^2-59a-36)} \end{bmatrix}$$

Numerical Example



Fuzzy total output for Agricultural Sector ($\bar{x}_1$, above) and Manufacturing Sector ($\bar{x}_2$, below) [1].

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