Final Draft

“Applied Fuzzy Linear Algebra: Input Output Analysis”

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1 Introduction

Linear systems of equations and there use in the analysis of the Unites States economy formed the foundations of a body of work that won Wassily Leontief a Nobel prize in Economics. He developed the Leontief input output (LIO) model. This model came in the mid-twentieth century and it was born out of the “high concentration of theory without fact on the one hand, and a mounting accumulation of fact without theory on the other.” [8]. Prior to the 1950s, most economists relied on their “professional opinion and sound judgment” to connect the mounting body of empirical facts with the firm foundations of economic theory.

Leontief sought to address this by imagining an economy where goods like wheat, iron, coal, etc. are produced in their respective sectors by means of a primary factor of production, typically labor, and by means other inputs such as coal, iron, alcohol, etc. [5] In many respects this is a natural idea because it makes intuitive sense; in order to grow wheat, one must have wheat to plant. So at the heart of Leontief input output model lies the idea that sectors of the economy are interrelated. The input output model is designed to quantify the relationship between the different industries or sectors of an economy. Out of this comes input output analysis, the method of systematically quantifying the mutual interrelationships among the various sectors of a complex economic system [8]. In Leontief’s original work on the input output model, he relied on data provided to him by the U.S. government. Since its inception as a economic model, it has often relied on empirical data which we learn in basic statistics often has uncertainties associated with even the best estimates.

It is unreasonable to assume the the Bureau of Economic Analysis or the Bureau of Labor Statistics are able to count every unit of production that is to come from any given sector. That job would be too expensive and time consuming. This is where fuzzy numbers can play a role in the Leontief input output model. Fuzzy numbers allow us to incorporate that uncertainty into our model and this will be the aim of this paper.

In Section Two we will build the necessary foundation for a basic understanding of fuzzy arithmetic. Section Three will cover the essentials to understanding the Leontief input output model. To that end we will cover the open model, first from a demand perspective and then from the perspective of prices and value added. Section four will examine solutions to a fuzzy input output model. To facilitate that effort, the required theorem, showing the existence of a fuzzy solution is presented. We then show a numerical example of a drastically simplified economy with two sectors. We will look at this example as a open model, closed model and a dynamic model.

2 Fuzzy Numbers

Fuzzy numbers as a field came out of the seminal work of Lotfi Zadeh in 1965 since then many authors have contributed to fuzzy set theory. In Arnold Kaufmann and Madan Gupta’s work, they outline the relationship between fuzzy numbers and intervals of confidence. A fuzzy number in considered an extension of the interval of confidence. Rather than looking at one level of confidence, as with the interval of confidence, fuzzy numbers consider intervals of confidence at all levels from 0 to 1. The level of presumption α, where $\alpha \in [0, 1]$, gives the interval of confidence $A^\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$, which is a monotonic decreasing function of α,

$$(\alpha' > \alpha) \Rightarrow ([a_1^{(\alpha')}, a_2^{(\alpha')}] \subset [a_1^{(\alpha)}, a_2^{(\alpha)}]),$$

for every $\alpha, \alpha' \in [0,1]$. This is to say as α increases, the interval of confidence never increases. In this manner, the level of presumption is linked to the interval of confidence. The coupling of these two concepts, the level α of presumption, and the interval of confidence at level α, will be the manner in which we use the concept of a fuzzy number [7].

A fuzzy number is a fuzzy subset where a membership function is defined for each element of the referential set [7]. Notationally we will borrow from [1,2,3] and denote a fuzzy number or fuzzy subset as a symbol with a bar. The referential set may be the integers, the real numbers, for our purposes the referential set will be the real numbers, $\mathbb{R}$. The membership function takes elements of $\mathbb{R}$ and maps them to the interval $[0,1]$,

$$\mu_{\bar{A}}(x) \in [0,1], \qquad \text{for all } x \in \mathbb{R}.$$

A fuzzy subset $\bar{A} \subset \mathbb{R}$ is normal if and only if,

$$\bigvee_x \mu_{\bar{A}}(x) = 1, \qquad \text{for all } x \in \mathbb{R}.$$

Which says that the highest (maximum) value of $\mu_{\bar{A}}(x)$ is equal to 1.

Convexity is defined as follows. If $\bar{A}$ is a fuzzy subset of $\mathbb{R}$ with membership function $\mu_{\bar{A}}(x)$, then $\bar{A}$ is convex if and only if every ordinary subset is convex,

$$A^\alpha = \{x | \mu_{\bar{A}}(x) \geq \alpha\}, \qquad \alpha \in [0,1];$$

that is if it is a closed interval of $\mathbb{R}$ [Gupta].

Finally a fuzzy number, $\bar{A}$, in $\mathbb{R}$, with membership function $\mu_{\bar{A}}(x)$, is a fuzzy subset of $\mathbb{R}$ that is both convex and normal.

If $\bar{A}$ and $\bar{B}$ are fuzzy numbers in $\mathbb{R}$ and we want to add, subtract, multiply or dived them, then we form their intervals of confidence for the level of presumption $\alpha \in [0,1]$ and define addition to be;

$$A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}].$$

We define subtraction of two fuzzy numbers to be,

$$\bar{A}(-)\bar{B} = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}],$$

which is equivalent to the addition of the image of $\bar{B}^-$ to A where $\bar{B}^- = [-b_2^{(\alpha)}, -b_1^{(\alpha)}]$. For definitions of the multiplication or the division of two fuzzy numbers refer to Appendix B of [Gupta].

A trapezoidal fuzzy number can be completely described by a quadruplet $\bar{A} = (a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$, and whose membership function is defined to be,

$$\mu_{\bar{A}}(x) \begin{cases} = \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ = 1, & a_2 \leq x \leq a_3 \\ = \frac{a_3-x}{a_3-a_2}, & a_3 \leq x \leq a_4 \\ = 0, & \quad x \geq a_3. \end{cases}$$

This means that the membership function is zero outside (a_1, a_4) and equals one on $[a_2, a_3]$ The graph of $y = \mu_{\bar{A}}(x)$ is monotonically increasing from zero to one on $[a_1, a_2]$; and the graph of $y = \mu_{\bar{A}}(x)$ is monotonically decreasing from one to zero on $[a_3, a_4]$. This is illustrated in figure 1.

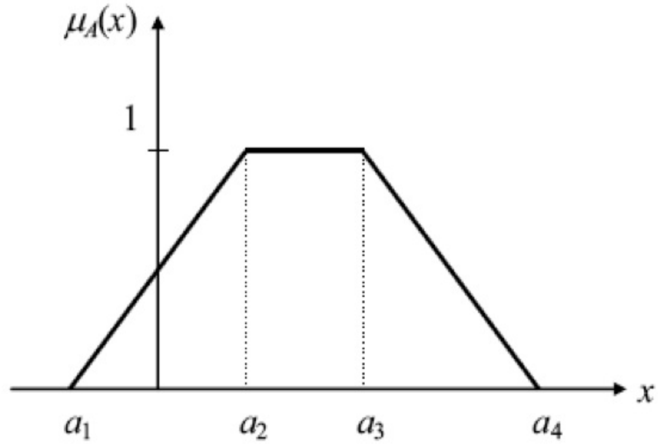

Figure 1: A trapezoidal fuzzy number $\bar{A} = (a_1, a_2, a_3, a_4)$

At the α level, the interval of confidence is given by,

$$\begin{aligned} A_\alpha &= [a_1^{(\alpha)}, a_4^{(\alpha)}] \\ &= [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]. \end{aligned}$$

The graph of $\mu_{\bar{A}}$ is trapezoidal when $a_2 < a_3$. When $a_2 = a_3 = a$, we write $\bar{A} = (a_1, a, a_4)$ and the graphs of $\mu_{\bar{A}}$ is triangular in shape. If $a_1 = a_2 = a_3 = a_4 = a$ then $\bar{A} = a$ is a crisp number [1,2,3] and can be written as $\bar{A} = (a, a, a, a)$.

3 Leontief input output Model

A number of assumptions are made in the input output model. Drawing from Fundamental Methods of Mathematical Economics, the assumptions that are typically made are: (a) Each industry produces only one homogenous good. (b) Each industry uses a fixed input ratio (referred to as the industry's technology) for the production of its output. (c) Production in every industry is subject to constant returns to scale so, for example, if we double the quantity of every input, we double the output. From here we move to the notation that is typically used when looking at a LIO model.

Suppose an economy can be divided in to n distinct sectors and using the notation from [Hands, Dorfman et. al. and Chiang], let a_{ij} be the amount of good i used in the production of one unit of good j where $i = 0, 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, a_{0j} represents the labor allocated to the production of the jth good's output. The a_{ij} are called input coefficients and they form the entries in $A = [a_{ij}]$, which is the input coefficient matrix or the technological matrix. Each entry in A is a nonnegative number and assumed to be constant, and for our purposes, we may assume prices to be given and adopt "a dollar's worth" of each good as its unit. If $a_{13} = .40$ this says that 40 cents' worth of the first produced good is required as an input for producing a dollar's worth of the third good. If x_j is the amount of good j actually produced, then the product $a_{ij}x_j$ represents the demand for food i and an input into the production of good j. When we sum over n, $\sum_{j=1}^{n} a_{ij}x_j$, this gives the total demand for good i as an intermediate good, that is the total demand for good i as an input to all industries. We also let final consumption for the output of each industry to be denoted by c_i where $i = 0, 1, 2, \ldots, n$, with $c_0 = 0$ for conventions sake. These values for final consumption form the entries for the vector $\mathbf{C}$, the final consumption vector. Finally let x_i be the total output of the ith good where $i = 0, 1, 2, \ldots, n$ and let x_0 be the total "production" of a primary non-produced good, in this case labor. These x_i's form the entires of the total output vector $\mathbf{X}$ whose entries denote the gross production of the ith good.

input output Table

	Output						
Input	Sector 1	Sector 2	...	Sector n	Households	Final Consumption ($\mathbf{C}$)	Total Output ($\mathbf{X}$)
Labor	a_{01}	a_{02}	...	a_{0n}	0	-	x_0
Sector 1	a_{11}	a_{12}	...	a_{1n}	a_{01}	c_1	x_1
Sector 2	a_{21}	a_{22}	...	a_{2n}	a_{02}	c_2	x_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sector n	a_{n1}	a_{n2}	...	a_{nn}	a_{0n}	c_n	x_n

3.1 The Leontief Open Model; Final Demand Perspective

The open Leontief model contains an "open" sector (often thought of as households) which exogenously determines the level of final consumption (the $\mathbf{C}$ vector) for the output of each industry and which supplies a primary input (here it is labor) that is not produced by the n industries themselves [4]. It should be noted that when looking at the open model, the sum of the entires in each column of the input-coefficient matrix must be less than one. That is $\sum_{i=1}^{n} a_{ij} < 1$ for $1 \leq j \leq n$. Each column sum represents the partial input cost (not including the cost of the primary input), of producing one dollar's worth of some commodity. If the $\sum_{i=1}^{n} a_{ij} > 1$, then this says that production is not economically justifiable because the partial input cost of producing one dollar's worth of the jth commodity is greater than one dollar [4].

When we look at an economy and use the open model to describe it we get the following equation,

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + c_i, \tag{1}$$

which tells us that the total production of the ith good is equal to the sum of the final consumption demand and the demand for the ith good as an input to the jth industry [6]. (1) can be rewritten as,

$$\mathbf{X} = A\mathbf{X} + \mathbf{C}, \tag{2}$$

where A is an $n \times n$ matrix, $\mathbf{X}$ is a $n \times 1$ column vector whose entries denote total output for each industry, and $\mathbf{C}$ is a $n \times 1$ column vector whose entries denote the exogenous demand or final consumption demand. When looking at the open model, the question that is typically asked is when, given a final consumption level for each good and each industry's technology, what level of gross output is needed to satisfy this final demand. That is if we are given a $\mathbf{C}$ and A we are seeking to find the $\mathbf{X}$ that satisfies (2). To this end, (2) can be rearranged to give $\mathbf{X} = (I - A)^{-1}\mathbf{C}$. But under what conditions are we guaranteed to have inverse, $(I - A)^{-1}$, that makes economic sense, that $[A_{ij}] \geq 0$ for all i and j?

This question is resolved when we look to Theorem 6.3 in [6], here we present it as Theorem 3.1.

Theorem 3.1. *For any $n \times n$ matrix B with $b_{ij} \leq 0$ for all $i \neq j$, the following three conditions are equivalent:*

(a) there exists an $\mathbf{x} \in \mathbb{R}^n_+$, such that $B\mathbf{x} > 0$

(b) B is a P matrix, that is B has strictly positive principal minors. A principal minor is the determinate of a submatrix formed from by "deleting" the same rows and columns form B. This is called a principal submatrix.

(c) $B^{-1} \geq 0$, that is $b_{ij} \geq 0$ for all i and j.

Proof. Consider an arbitrary input coefficient matrix G. Define $B = I - G$ so $b_{ii} = 1 - g_{ii}$ for all i and $b_{ij} = -g_{ij}$ for all $i \neq j$. Since $g_{ij} \geq 0$ for all $i \neq j$, matrix B has $b_{ij} \leq 0$ for all $i \neq j$. Thus B satisfies the first condition for the application of the above theorem. Now suppose that the Leontief matrix G is productive in the following sense. Suppose there exists a nonnegative n-dimensional column vector $\tilde{\mathbf{x}}$ such that

$$\tilde{\mathbf{x}} > G\tilde{\mathbf{x}}, \tag{3}$$

which says that there exists a set of output levels $\tilde{x}_i \geq 0$ such that $\tilde{x}_i > \sum_{j=1}^{n} a_{ij}x_j$. This tells us that there is at least one way to run an economy "productively", that there is a way to produce more of each good than is consumed as intermediate inputs. From this condition, it follows that there is a strictly positive final consumption vector. Note that if we let each entry in the vector $\tilde{x}$ be equal to one, $\tilde{x}_i = 1$, then $1 > \sum_{j=1}^{n} a_{ij}$ which is an extension of the idea that an economically feasible economy is synonymous with a productive economy. Any system that is seen to be producing a positive output of all goods automatically satisfies condition (3).

When an economy is productive in the sense of (3), the matrix $B = I - G$ satisfies part (a) of Theorem 3. Since B is a P matrix this guarantees that B is non-singular and finally part (c) implies that $B^{-1} = (I - G)^{-1} \geq 0$. Therefore the productivity condition (3) guarantees that the open LIO system as a strictly positive total output vector $\mathbf{X} = (I - A)^{-1}\mathbf{C}$ for any final consumption vector $\mathbf{C} > 0$ □

3.2 The Leontief Open Model; Price Perspective

We have been looking at the open Leontief input output model from the demand side yet there is a price perspective as well. Let us consider prices, by letting $\mathbf{p} = (p_1, p_2, \ldots, p_n) > 0$

be a $1 \times n$ row vector of prices for the n outputs. With these prices, $\sum_{i=1}^{n} p_i a_{ij}$, is the total cost of the intermediate goods necessary to produce one unit's worth of good j. From here we can define the idea of value added. The value added per unit of j is the price of one unit of good j minus the total cost of inputs. Mathematically this statement is,

$$V_j = p_j - \sum_{i=1}^{n} p_i a_{ij}, \tag{4}$$

and can be rewritten as a vector matrix product, $\mathbf{V} = \mathbf{p}(I - A)$ where $\mathbf{V}$ is a $1 \times n$ row vector whose entries, v_j, are the value added per unit of the jth good, $\mathbf{p}$ is the $1 \times n$ price row vector defined above, I is the $n \times n$ identity matrix and A is an $n \times n$ Leontief input output matrix [6]. Similar to Section 3.1, we ask if there is an economically significant solution to $\mathbf{p} = \mathbf{V}(I - A)^{-1}$ for any given value added vector $\mathbf{V}$ where $v_j > 0$ for all j.

The answer is yes if we know that the system is productive in the same sense as (3). But (3) is an output condition and if we do not know wether a system satisfies the productivity condition (3) then we still need to find a satisfactory answer [6]. To find a price analog to the productivity condition (3) we look to see if a system is profitable. If there exists a price vector whose entries are non-negative such that there is a strictly positive value added vector, $v_j > 0$, then $(I - A)^{-1}$ and positive prices exist for any initial value-added vector [6]. A consequence of this is that if the system is profitable then there exists $\tilde{\mathbf{p}}0$ such that

$$\tilde{\mathbf{p}} > A\tilde{\mathbf{p}}. \tag{5}$$

The proof of this may be found in [6] and is based off of Theorem 6.4 in [6].

Now in long-run competitive equilibrium we may put prices equal to unit costs. If a good is produced at all, the equality must hold, but prices may fall short of unit cost for a commodity no being produced; this is why the good is not being produced [5]. Now we recall that we defined a_{0j} represents the labor allocated to the production of one unit of the jth sector's output. If we designate a wage rate by w then the product, wa_{0j} tells us the labor cost associated with the production of one unit of good j. With only one primary input, labor, In long-run equilibrium with zero economic profit, the value added in each sector will be paid to the source of the primary input, here the value added will be paid to the households, $wa_{0j} = v_j$. When this is the case,

$$w\mathbf{a_0} = p(I - A)^{-1} \tag{6}$$

where $\mathbf{a_0}$ is the column vector whose entries are a_{0j}. By looking at the long run equilibrium relationship in (5) we can find the wage that needs to be paid to the owners of the primary input if given a technology matrix and a price vector. And because of the profitability condition (5) we can expect there to be an economically meaningful wage.

4 Solving Fuzzy Leontief Systems

Now that we have a basic understanding of the LIO model and fuzzy numbers, combining them will require some notation. $\bar{A} = [\bar{a}_{ij}]$ is a $n \times n$ matrix of fuzzy numbers $\bar{a}_{ij} =$

$(a_{ij1}|a_{ij2}, a_{ij3}|a_{ij4})$ where $0 \leq a_{ij1} \leq a_{ij2} \leq a_{ij3} \leq a_{ij4} \leq 1$ represent the fuzzy input coefficients. Next let $\bar{\mathbf{C}} = [\bar{c}_i]$ be an $n \times 1$ vector where $\bar{c}_i = (c_{i1}|c_{i2}, c_{i3}|c_{i4})$ and $\bar{c}_i$ is non-negative. $\bar{\mathbf{C}}$ is a fuzzy vector of final consumption demand. Finally, let $\bar{\mathbf{X}} = [\bar{x}_i]$ be an $n \times 1$ where $\bar{x}_i = (x_{i1}|x_{i2}, x_{i3}|x_{i4})$ and $\bar{x}_i$ is non-negative. $\bar{\mathbf{X}}$ is a fuzzy vector of total output for industries in this economy [1,2,3].

With this notation we can incorporate fuzzy numbers into the LIO model. When looking at the open LIO model, equation (2) becomes,

$$\bar{\mathbf{X}} = \bar{A}\bar{\mathbf{X}} + \bar{\mathbf{C}} \tag{7}$$

with rearranging in the same manner as in Section 3.1,

$$\bar{\mathbf{X}} = (I - \bar{A})^{-1}\bar{\mathbf{C}}, \tag{8}$$

allows us to ask the same question as we posed then. With a fuzzy open LIO model we can ask what level total output will ensure a given final consumption demand with a given technology? To answer this, we apply standard fuzzy addition and multiplication as outlined in Section 2 of this paper.

Fuzzy arithmetic is more easily preformed in terms of intervals of confidence for the level of presumption $\alpha \in [0, 1]$, called α-cuts. Following the Buckley's method he outlined in "Fuzzy input output Analysis" we define,

$$\begin{aligned}\bar{a}^{\alpha}_{ij} &= [a^{\alpha}_{ijl}, a^{\alpha}_{iju}]\\ \bar{c}^{\alpha}_{i} &= [c^{\alpha}_{il}, c^{\alpha}_{iu}]\\ \bar{x}^{\alpha}_{i} &= [x^{\alpha}_{il}, x^{\alpha}_{iu}]\end{aligned}$$

Next we set $\bar{A}^{\alpha}_l = [a^{\alpha}_{ijl}], \bar{A}^{\alpha}_u = [a^{\alpha}_{iju}], \bar{\mathbf{C}}^{\alpha}_l = [c^{\alpha}_{il}], \bar{\mathbf{C}}^{\alpha}_u = [c^{\alpha}_{iu}], \bar{\mathbf{X}}^{\alpha}_l = [x^{\alpha}_{il}]$ and $\bar{\mathbf{X}}^{\alpha}_u = [x^{\alpha}_{iu}]$. Once this is done, fuzzy arithmetic based on α-cut arithmetic becomes the interval arithmetic outlined above and equation (8) is equal to

$$\mathbf{X}^{\alpha}_l = (I - A^{\alpha}_l)^{-1}\mathbf{C}^{\alpha}_l, \tag{9}$$
$$\mathbf{X}^{\alpha}_u = (I - A^{\alpha}_u)^{-1}\mathbf{C}^{\alpha}_u, \tag{10}$$

provided the inverse exists. But how do we know if there is a $\bar{X}$ for any (non-negative) fuzzy final consumption demands, $\bar{\mathbf{C}}$? Buckley offers the following theorem that addresses this question.

Theorem 4.1. *If $\sum_{i=1}^{n} a_{ij4} < 0$ for all j, then the fuzzy input output model exists for this economy.*

Proof. We will be following [1] for this proof. Let $W = A^{\alpha}_l$ or A^{α}_u for any level of presumption, $\alpha \in [0, 1]$. We know that the column sums of W are less than one and that W is a semi-positive square matrix. Let $\mathbf{v} = (1, \ldots, 1)$ be a $1 \times n$ vector whose elements are all equal to one. Then the vector-matrix product,

$$\mathbf{v}W = \mathbf{s} = (s_1, \ldots, s_m) \tag{11}$$

where $0 \leq s_i \leq 1$ for all $0 \leq i \leq 1$. We know that W has a particular eigenvalue, λ^* with associated eigenvector $\mathbf{x}^*$ such that:

1. λ^* is real and non-negative;
2. no other eigenvalue had modulus exceeding λ^*, that is $|\lambda| < \lambda^*$;
3. $\mathbf{x}^*$ is non-negative; and
4. for any real number, $z > \lambda^*, zI - W$ is nonsingular and its inverse is semi-positive, meaning that the inverse has only real numbers greater than or equal to zero and no row or column is entirely composed of zeros.

From here we argue that $\lambda^* < 1$ which implies that $I - A_l^\alpha$ and $I - A_l^\alpha$ are both nonsingular and their inverses are semi-positive. We know by the definition of an eigenvalue that:

$$W\mathbf{x}^* = \lambda^* \mathbf{x}^* \tag{12}$$

so that

$$\mathbf{v}W\mathbf{x}^* = \lambda^*(\mathbf{v}\mathbf{x}^*). \tag{13}$$

Therefore since we know that the entires in $\mathbf{s}$ are in the closed interval $[0, 1]$, that is $0 < s_i < 1$ for all i,

$$\mathbf{s}\mathbf{x}^* = \lambda^* \sum_{i=1}^{n} x_i^*. \tag{14}$$

It follows that

$$\lambda^* \sum_{i=1}^{n} x_i^* < \sum_{i=1}^{n} x_i^*. \tag{15}$$

and $\lambda^* < 1$. Equation (15) tells us that when we assume that the column sums of the fuzzy input coefficient matrix are less than one, we can expect $(I - A_l^\alpha)^{-1}$ and $(I - A_u^\alpha)^{-1}$ to exist. □

Note the similarity between the conclusion in Theorem 3.1 that requires all of the entires in a crisp technology matrix to be positive and strictly less than one and the conclusion that we reach with (15). Theorem 4.1 is an intuitive extension of Theorem 3.1 to a fuzzy input coefficient matrix. If the partial input costs of producing one dollar's worth of the jth good exceeds one dollar, it is not economically justifiable to produce the jth commodity.

Recall the similarity between the productivity condition (4) and profitability condition (5) outlined in Sections 3.1 and 3.2 respectively. When we see the duality between productivity and profitability with crisp values in addition to recognizing the connection between the conclusions reached in Theorems 3.1 and 4.1 it is not a far leap to suspect that there must be a profitability analog with fuzzy input coefficients. However, we will simply acknowledge the existence of a profitability analog when dealing with fuzzy input output values.

5 Numerical Example

We will use the following example of a two sector economy (Agriculture and Manufacturing) with a primary input of labor to illustrate the application of the Open LIO model, the Closed LIO model and the Dynamic LIO model. The follow table shows the fuzzy input coefficients and the fuzzy final consumption vector for a simple two sector economy.

Numerical Example

	Industries			
	Agriculture	Manufacturing	Final Consumption ($\bar{\mathbf{C}}$)	Gross Output ($\bar{\mathbf{X}}$)
Agriculture	$(0.25/0.3/0.35)$	$(0.3/0.4/0.5)$	$(60/65, 75/80)$	$\bar{x}_1$
Manufacturing	$(0.4/0.45, 0.55/0.60)$	$(0.2/0.25, 0.35/0.4)$	$(50/55, 65/70)$	$\bar{x}_2$
Outside Inputs	$(0.1/0.2/0.3)$	$(0.2/0.3/0.4)$		
Total	$(0.75/0.95, 1.05/1.25)$	$(0.7/0.95, 1.05/1.3)$		

The information in this table says that for example, the for the agriculture sector to produce one dollar's worth of output, it needs around 30 cents worth of inputes from the agriculture sector while the agriculture sector needs between 45 cents and 55 cents worth of inputs from the manufacturing sector.

Starting with the Open LIO model we first check the condition in theorem 1. Since $a_{114}+a_{214} < 1$ and $a_{214}+a_{224} < 1$ the condition of Theorem 2 is met so we are guaranteed a solution to $\bar{x}_1$ and $\bar{x}_2$. To simplify the following computations, we assume that the membership function for $\bar{a}_{ij}$ is a straight line on $[a_{ij1}, a_{ij2}]$ and on $[a_{ij3}, a_{ij4}]$ for all i and j. We make similar assumptions about $\bar{c}_i$ and $\bar{x}_i$.

Since we know a solution exits to this fuzzy LIO system, we form the α-cuts for each a_{ij} and c_i respectively. This results in two systems of linear equations.

$$\bar{\mathbf{X}}_l^\alpha = \begin{bmatrix} x_{1l}^\alpha \\ x_{2l}^\alpha \end{bmatrix}, \bar{A}_l^\alpha = \begin{bmatrix} 0.25+0.05\alpha & 0.4+0.1\alpha \\ 0.4+0.05\alpha & 0.2+0.05\alpha \end{bmatrix}, \bar{\mathbf{C}}_l^\alpha = \begin{bmatrix} 60+5\alpha \\ 50+5\alpha \end{bmatrix}$$

$$\bar{\mathbf{X}}_u^\alpha = \begin{bmatrix} x_{1u}^\alpha \\ x_{2u}^\alpha \end{bmatrix}, \bar{A}_u^\alpha = \begin{bmatrix} 0.35-0.05\alpha & 0.5-0.1\alpha \\ 0.6-0.05\alpha & 0.4-0.05\alpha \end{bmatrix}, \bar{\mathbf{C}}_u^\alpha = \begin{bmatrix} 80-5\alpha \\ 70-5\alpha \end{bmatrix}$$

Solving equations (6) and (7) produces the fuzzy numbers $\bar{x}_1$ and $\bar{x}_2$ whose graphs can be seen in Figure 2.

This tells us that the the level of total output needed from each sector of our simple economy that will satisfy a given fuzzy final demand is for the agricultural sector, depending on the level of presumption, is described in the top graph on Figure 2. The bottom graph describes the fuzzy total manufacturing output that is needed to satisfy the fuzzy exogenous demand.

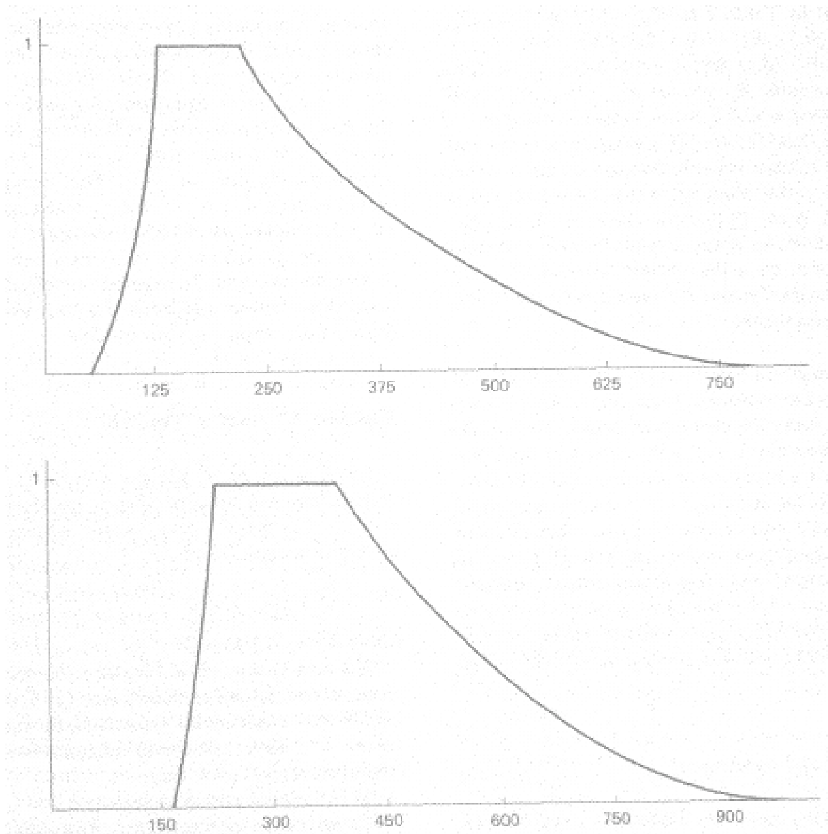

Figure 2: Fuzzy total output for Agricultural Sector ($\bar{x}_1$, above) and Manufacturing Sector ($\bar{x}_2$, below) [1].

6 Conclusion

In this paper we have looked at a common application of linear algebra. The relationship between linear systems and economics is where we have focused our attention. We have examined the open Leontief input output model and looked into how to find levels of output that would satisfy a given level of exogenous final demand when we have a crisp technology matrix, a crisp total output vector and a crisp final demand vector. More interestingly and more realistic we consider that much of the data that describes the economy has some level of uncertainty associated with it. In so doing we pulled from J.J Buckley's work on fuzzy input output analysis. The fuzzy Leontief model which allows us to incorporate the uncertainty or fuzziness of the data in the from of fuzzy triangular numbers into our model formed the basis for much of this paper. We provided a drastically simplified example of an economy with only two sectors and we looked at what the equilibrium total output would be with a given fuzzy final demand and fuzzy technology matrix as well as the fuzzy price vector.

Fuzzy input output analysis has been extended to the closed Leontief model and in the same manner to the dynamic Leontief model. This is an area that could be studied further and would prove interesting to a linear algebra student.

7 Sources

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