General Inner Product and The Fourier Series

A Linear Algebra Approach

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4-20-14 / Spring Semester

Outline

Inner Product

- The inner product is an algebraic operation that takes two vectors and computes a single number, a scalar.
- Introduces a geometric intuition for length and angles of vectors.
- Generalization of Dot product
- Euclidean space and Hilbert Space

Inner Products

- **Definition** Real Inner Product
  Let V be a real vector space and $a, b \in V$. An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{R}$ satisfying the following conditions:
  - $\langle \alpha a + \alpha' b, c\rangle = \alpha\langle a, c\rangle + \alpha'\langle b, c\rangle$
  - $\langle c, \alpha a + \alpha' b\rangle = \alpha\langle c, a\rangle + \alpha'\langle c, b\rangle$
  - $\langle a, b\rangle = \langle b, a\rangle$
  - $\langle a, a\rangle$ is a positive real number for any $a \neq 0$
- **Definition** Complex Inner Product
  Let V be a real vector space and $a, b \in V$. An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{C}$ satisfying the following conditions:
  - $\langle \alpha a + \alpha' b, c\rangle = \alpha\langle a, c\rangle + \alpha'\langle b, c\rangle$
  - $\langle c, \alpha a + \alpha' b\rangle = \alpha^*\langle c, a\rangle + \alpha'^*\langle c, b\rangle$
  - $\langle a, b\rangle = \overline{\langle b, a\rangle}$
  - $\langle a, a\rangle$ is a positive real number for any $a \neq 0$

Inner Products

- The complex inner product is referred to as sesquilinear or hermitian because it is linear in one term (coordinate) and antilinear or conjugate-linear or semilinear in the other term.
- Dot product is bilinear.

Inner Products

- **Definition** Orthogonal Vectors
  If V is a vector space, $a, b \in V$ and $\langle a, b \rangle = 0$ then a and b are orthogonal to eachother.
- **Cauchy-Shwarz-Bunyakovsky inequality**
  If V is a vector space and $a, b \in V$ then,

  $$|\langle b, c \rangle| \leq ||b|| \cdot ||c||$$

  where,

  $$||b|| = \sqrt{\langle b, b \rangle}$$

  and $||b||$ is the length of b.

Inner Products

- **Definition** Distance
  Let V be a vector space. Then the distance between any two vectors $a, b \in V$ is defined by,
  $$d(a, b) = ||a - b||$$
- **Theorem** Distance Corollary
  Let V be a vector space. Then the distance between any two factors $a, b \in V$ satisfies,
  $d(b, c) \geq 0$ for any two vectors unless $b = c$ then $d(b, c) = 0$
  $$d(b, c) = d(c, b) \text{ (symmetry)}$$

Inner Products

- **Bessel's Inequality**
  Suppose V is a vector space and $B = \{u_1, ..., u_n\}$ is an orthonormal basis for V. Then for any $v \in V$,
  $$||v||^2 \geq |\langle v, u_1 \rangle|^2 + ... + |\langle v, u_n \rangle|^2$$
- **Theorem** Inner Product Equivalence
  Suppose V is a vector space with orthonormal basis $B = \{u_1, ..., u_k\}$. Then any vector $v \in V$ has the following equivalences,
  $$v = \sum_{k=1}^{n} \langle v, u_k \rangle u_k$$
  $$||v||^2 = \sum_{k=1}^{n} |\langle v, u_k \rangle|^2$$

Inner Products

- **Definition** Closed System
  If V is a vector space and $B = \{u_1, ..., u_k\}$ is an orthonormal basis of V then $\{u_1, ..., u_k\}$ is a closed system if for every $v \in V$,
  $$v = \sum_{k=1}^{\infty} \langle v, u_k \rangle u_k$$
  That is, if the sequence $\{v_n\}$ defined by,
  $$v_n = \sum_{k=1}^{n} \langle v, u_k \rangle u_k$$
  converges to v,
  $$||v - v_n|| \to 0$$
  as,
  $$n \to \infty$$

Fourier Series

- Expansion of periodic function
- Infinite sines and cosines
- Harmonic Analysis
- Breaking up function into set of terms
- Recombination of terms to offer solution or approximation

Fourier Series

- **Theorem** Fourier Expansion
  Suppose V is a vector space and $B = \{u_1, ..., u_n\}$ is an orthonormal basis for V. Then for any $v \in V$,

  $$v = \langle v, u_1 \rangle u_1 + ... + \langle v, u_n \rangle u_n$$

  Where the $\langle v, u_i \rangle$ coordinates are the Fourier coefficients of v with respect to B and $\langle v, u_1 \rangle u_1 + ... + \langle v, u_n \rangle u_n$ is the Fourier expansion of v with respect to B.
- **Theorem** Inner Product of continuous functions over the complex numbers

  $$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

Fourier Series

- **Definition** Real form of Fourier Series
  Suppose we have a periodic function $f(x)$ with a period of 2π , where $k = 0, 1, ...$ Then the series is defined as,

$$a_o/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where the coefficients a_k, b_k are defined as,

$$a_k = 1/\pi \int_{-\pi}^{\pi} f(x) \cos kx dx$$

$$b_k = 1/\pi \int_{-\pi}^{\pi} f(x) \sin kx dx$$

Fourier Series

- **Definition** Complex form of Fourier Series
  Suppose we have a periodic function $f(x)$ with a period of 2π, where $k = 0, 1, ...$ Then the series is defined as,

$$\sum_{-\infty}^{\infty} c_k e^{ikx}$$

  where c_k is defined as,

$$c_k = 1/2\pi \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

L^2 Space

- Consists of equivalence functions
- Same L^2 function if difference in sets of functions measures to zero
- $([-\pi, \pi])$
- Natural space for periodic functions
- Wave functions

L^2 Space

- **Definition** L^2 $([-\pi, \pi])$ Space
  L^2 is the set of all complex-valued functions on $[-\pi, \pi]$ that satisfy,
  $$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$$
  Where the inner product is,
  $$\langle f, g \rangle = 1/\pi \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx$$

Fourier Series

- **Theorem** Fourier Series of $L^2\left([-\pi,\pi]\right)$
  If $f \in L^2$ then its Fourier Series is,

$$a_o/2 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

  where,

$$a_k = \langle f, \cos kt \rangle$$

$$b_k = \langle f, \sin kt \rangle$$

Fourier Series

- **Definition** Convergence
  Suppose we have a sequence of vectors (v_n) in an inner product space. Then the set converges to $v \in V$ if:

$$\lim_{n\to\infty} \mathrm{d}(v_n, v) = 0$$

or,

$$\lim_{n\to\infty} ||v_n - v|| = 0$$

Fourier Series

- **Theorem** Convergence of Fourier Series
  Let $f \in L^2$ and set,

$$g(t) = a_o/2 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt)$$

Then $g(t) \in L^2$ and,

$$||g(t) - f||^2 = 1/\pi \int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt \to 0$$

as $n \to 0$

- **Theorem** Parseval's
  Let V be a vector space and $B = \{u_1, ..., u_k\}$ be a closed orthonormal basis in V. Then for any $v, w \in V$,

$$\langle v, w \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$$

where,

$$a_k = \langle v, u_k \rangle, b_k = \langle w, u_k \rangle$$

and,

$$||v||^2 = \sum_{k=1}^{\infty} |\langle v, u_k \rangle|^2 = \sum_{k=1}^{\infty} |a_k|^2$$

Example

The input to an electrical circuit that switches between a high and a low state with time period 2π can be represented by the boxcar function,

$$f(x) = 1 \text{ when } 0 \leq x \leq \pi$$

$$f(x) = -1 \text{ when } -\pi \leq x \leq 0$$

The periodic expansion of this function is referred to as the square wave function. Generally this is the input to an electrical circuit that switches from a high to low state with time period T which can be represented by the general square wave function with the basic period,

$$f(x) = 1 \text{ when } 0 \leq x \leq T/2$$

$$f(x) = -1 \text{ when } -T/2 \leq x \leq 0$$

Example

$$
\begin{aligned}
b_k &= 1/\pi \int_{-\pi}^{\pi} f(x) \sin kx dx \\
&= 2/\pi \int_{0}^{\pi} f(x) \sin kx dx \\
&= -2/k\pi \cos kx|_0^\pi \\
&= -2/k\pi((-1)^k - 1)
\end{aligned}
$$

Notice that $((-1)^k - 1) = 1 - 1 = 0$ if k is even $(2k)$ but it is $= -2$ if k is odd $(2k+1)$.
Thus,

$$b_{2k} = 0 \text{ and } b_{2k+1} = -2/k\pi(-2) = 4/(2k+1)\pi$$

and we get,

Example

$$f(x) = 4/\pi \sum_{k=odd}^{\infty} (1/k) \sin kx$$

$$= 4/\pi \sum_{k=0}^{\infty} (\sin(2k+1)x)/2k+1$$

$$= 4/\pi(\sin x + \sin 3x/3 + \sin 5x/5 + ...)$$

For the representation of the general square wave we obtain $a_n = 0, b_{2k} = 0$ and thus,

$$b_{2k+1} = (4/(2k+1)\pi) \sin((2(2k+1)\pi x)/T)$$

and we get the final representation,

$$f(x) = 4/\pi \sum_{k=0}^{\infty} (1/2k+1) \sin((2(2k+1)\pi x)/T)$$

Summary

- Inner Product Intuition
  - Orthogonality
  - Closure
  - Convergence and Equivalence
  - Distance and Difference
- Fourier Series
  - Expansion
  - Orthogonality of sine and cosine
  - L^2 space and Functions