The Polar Decomposition SVD) and Polar Decomposition (Geometric Concepts	Applications	Conclusion

Polar Decomposition of a Matrix

Garrett Buffington

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The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications 00	Conclusion
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- 1 The Polar Decomposition
 - What is it?
 - Square Root Matrix
 - The Theorem

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5 Conclusion

The Polar Decomposition ●○○○○	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
What is it?				

Definition (Right Polar Decomposition)

The right polar decomposition of a matrix $A \in \mathbb{C}^{m \times n}$ $m \ge n$ has the form A = UP where $U \in \mathbb{C}^{m \times n}$ is a matrix with orthonormal columns and $P \in \mathbb{C}^{n \times n}$ is positive semi-definite.

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
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Definition (Right Polar Decomposition)

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Definition (Left Polar Decomposition)

The left polar decomposition of a matrix $A \in \mathbb{C}^{n \times m}$ $m \ge n$ has the form A = HU where $H \in \mathbb{C}^{n \times n}$ is positive semi-definite and $U \in \mathbb{C}^{n \times m}$ has orthonormal columns.

The Polar Decomposition ○●○○○	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion	
Square Root of a Matrix					

Theorem (The Square Root of a Matrix)

If A is a normal matrix then there exists a positive semi-definite matrix P such that $A = P^2$.

The Polar Decomposition ○●○○○	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
Square Root o	of a Matrix			

Theorem (The Square Root of a Matrix)

If A is a normal matrix then there exists a positive semi-definite matrix P such that $A = P^2$.

Proof.

Suppose you have a normal matrix A of size n. Then A is orthonormally diagonalizable. This means that there is a unitary matrix S and a diagonal matrix B whose diagonal entries are the eigenvalues of A so that $A = SBS^*$ where $S^*S = I_n$. Since A is normal the diagonal entries of B are all positive, making B positive semi-definite as well. Because B is diagonal with real, non-negative entries we can easily define a matrix C so that the diagonal entries of C are the square roots of the eigenvalues of A. This gives us the matrix equality $C^2 = B$. Define P with the equality $P = SCS^*$.

The Polar Decomposition ○○●○○	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
The Theorem				

Definition (P)

The matrix P is defined as $\sqrt{A^*A}$ where $A \in \mathbb{C}^{m \times n}$.

The Polar Decomposition ○○●○○	SVD and Polar Decomposition	Geometric Concepts	Applications 00	Conclusion
The Theorem				

Definition (P)

The matrix *P* is defined as $\sqrt{A^*A}$ where $A \in \mathbb{C}^{m \times n}$.

Theorem (Right Polar Decomposition)

For any matrix $A \in \mathbb{C}^{m \times n}$, where $m \ge n$, there is a matrix $U \in \mathbb{C}^{m \times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n \times n}$ so that A = UP.

The Polar Decomposition ○○○●○	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Example				

$$\begin{array}{c} A \\ A = \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix} A^* A = \begin{bmatrix} 14 & 38 & 26 \\ 38 & 105 & 75 \\ 25 & 76 & 89 \end{bmatrix}$$

The Polar Decomposition ○○○●○	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
Example				

$$\begin{array}{c} A \\ A = \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix} A^* A = \begin{bmatrix} 14 & 38 & 26 \\ 38 & 105 & 75 \\ 25 & 76 & 89 \end{bmatrix}$$

<i>S</i> , <i>S</i>	$^{-1}$, and C					
<i>S</i> =	$\begin{bmatrix} 1 \\ -0.3868 \\ 0.0339 \end{bmatrix}$	1 2.3196 -3.0376	1 2.8017 2.4687			
	$= \begin{bmatrix} 0.8690 \\ 0.0641 \\ 0.0669 \end{bmatrix}$	-0.3361	0.0294	4 16 2		
<i>C</i> =	0.4281	0 4.8132	0 0 .5886	-		

The Polar Decomposition ○○○○●	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
Example				
Р				

	[1.5897	3.1191	1.3206
$P = \sqrt{A^*A} = S^*CS^{-1} =$	3.1191	8.8526	4.1114
	1.3206	4.1114	8.3876

The Polar Decomposition ○○○○●	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
Example				

Ρ

	[1.5897	3.1191	1.3206]
$P = \sqrt{A^*A} = S^*CS^{-1} =$	3.1191	8.8526	4.1114
	1.3206	4.1114	8.3876

U			
	0.3019	0.9175	-0.2588
U =	0.6774	0.9175 -0.0154 0.3974	0.7355
		0.3974	0.6262

The Polar Decomposition ○○○○●	SVD and Polar Decomposition	Geometric Concepts	Applications 00	Conclusion
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U

	0.3019	0.9175	-0.2588 0.7355 0.6262]	
U =	0.6774	-0.0154	0.7355	
	0.6708	0.3974	0.6262	

A							
	1.5897	3.1191	1.3206	0.3019	0.9175	-0.2588 0.7355 0.6262	
UP =	3.1191	8.8526	4.1114	0.6774	-0.0154	0.7355	
	1.3206	4.1114	8.3876	0.6708	0.3974	0.6262	

The Polar Decomposition	SVD and Polar Decomposition ●○○	Geometric Concepts 00000	Applications 00	Conclusion
Polar Decom	osition from SV/F)		

Theorem (SVD to Polar Decomposition)

For any matrix $A \in \mathbb{C}^{m \times n}$, where $m \ge n$, there is a matrix $U \in \mathbb{C}^{m \times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n \times n}$ so that A = UP.

The Polar Decomposition	SVD and Polar Decomposition ●○○	Geometric Concepts 00000	Applications 00	Conclusion
Polar Decomp	osition from SVD)		

Theorem (SVD to Polar Decomposition)

For any matrix $A \in \mathbb{C}^{m \times n}$, where $m \ge n$, there is a matrix $U \in \mathbb{C}^{m \times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n \times n}$ so that A = UP.

Proof.

$$A = U_S SV^*$$

= $U_S I_n SV^*$
= $U_S V^* V SV^*$
= UP

The Polar Decomposition	SVD and Polar Decomposition ○●○	Geometric Concepts	Applications	Conclusion
Example Usin	g SVD			

Give Sage our A and ask to find the SVD



The Polar Decomposition	SVD and Polar Decomposition ○●○	Geometric Concepts	Applications 00	Conclusion
Example Usin	g SVD			

Give Sage our A and ask to find the SVD



Components

	0.5778	0.8142	0.0575
$U_S =$	0.6337	0.4031	0.6602
	0.5144	0.4179	0.7489
[13.5886	0	οĪ
<i>S</i> =	0	4.8132	0
	0	0	0.4281
-	0.2587	0.2531	0.9322
V =	0.7248	0.5871	0.3605
	0.6386	0.7689	0.0316

The Polar Decomposition	SVD and Polar Decomposition ○○●	Geometric Concepts 00000	Applications 00	Conclusion
Example Using	g SVD			

U	
$ \begin{array}{l} U = U_S V^* \\ = \begin{bmatrix} 0.5778 & 0.8142 & 0.0575 \\ 0.6337 & 0.4031 & 0.6602 \\ 0.5144 & 0.4179 & 0.7489 \end{bmatrix} \begin{bmatrix} -0.2 \\ 0.2 \\ -0.6 \\ -0.6 \end{bmatrix} \\ = \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix} $	531 0.5871 -0.7689

The Polar Decomposition	SVD and Polar Decomposition ○○●	Geometric Concepts 00000	Applications 00	Conclusion
Example Using	g SVD			

U				
$U = U_{S}V^{*}$ = $\begin{bmatrix} 0.5778\\ 0.6337 \end{bmatrix}$	0.8142 0.0575 0.4031 0.6602 0.2531	-0.7248 0.5871	-0.6386 -0.7689	
$= \begin{bmatrix} 0.6337 \\ 0.5144 \\ 0.3019 \end{bmatrix}$	$ \begin{array}{c ccccc} 0.4031 & 0.0002 \\ 0.4179 & 0.7489 \\ 0.9175 & -0.2588 \\ \end{array} $		-0.7689	
$= \begin{bmatrix} 0.6774 \\ -0.6708 \end{bmatrix}$	-0.0154 0.7355 0.3974 0.6262			

Р	
$\begin{split} P &= VSV^* \\ = \begin{bmatrix} 0.2587 & 0.2531 \\ 0.7248 & 0.5871 \\ 0.6386 & 0.7689 \\ 1.5897 & 3.1191 \\ 3.1191 & 8.8526 \\ 1.3206 & 4.1114 \\ \end{split}$	$ \begin{bmatrix} 0.9322\\ 0.3605\\ 0 & 4.8132 & 0\\ 0 & 0 & 0.4281 \end{bmatrix} \begin{bmatrix} -0.2587 & -0.7248 & -0.6386\\ 0.2531 & 0.5871 & -0.7689\\ -0.9322 & 0.3605 & -0.0316 \end{bmatrix} \\ \begin{bmatrix} 13.206\\ 4.1114\\ 8.3876 \end{bmatrix} $

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Geometry Cor	cents			

J

Matrices A = UP

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Geometry Cor	ncepts			

Matrices

J

A = UP



	000 00	
Motivating Example		

2×2			
$A = \begin{bmatrix} 1.300 \\ .750 \end{bmatrix}$	375 .650		

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts ●0000	Applications	Conclusion
Motivating Ex	ample			

$$2 \times 2$$

$$A = \begin{bmatrix} 1.300 & -.375 \\ .750 & .650 \end{bmatrix}$$

Polar Decomposition

$$U = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$$
$$P = \begin{bmatrix} 1.50 & 0.0 \\ 0.0 & 0.75 \end{bmatrix} = \sqrt{A^*A}$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications 00	Conclusion
P and r				

$$2 \times 2$$

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$r = \sqrt{x^2 + y^2}$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
P and r				

$$2 \times 2$$

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$r = \sqrt{x^2 + y^2}$$

r Vector

$$\|\mathbf{r}\| = \sqrt{\mathbf{r}^* \mathbf{r}}$$

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P and r				

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$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

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$$\|\mathbf{r}\| = \sqrt{\mathbf{r}^* \mathbf{r}}$$

Ρ

$$P = \sqrt{A^*A}$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion

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Continuum Mechanics

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion

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Continuum Mechanics

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Computer Graphics

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications ●○	Conclusion
Iterative Methods for U				

Newton Iteration

$$U_{k+1} = \frac{1}{2}(U_k + U_k^{-t}), \quad U_0 = A$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications ●○			
Iterative Methods for U						

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$$U_{k+1} = \frac{1}{2}(U_k + U_k^{-t}), \quad U_0 = A$$

Frobenius Norm Accelerator

$$\gamma_{F_k} = \frac{\|U_k^{-1}\|_F^{\frac{1}{2}}}{\|U_k\|_F^{\frac{1}{2}}}$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications ●○	Conclusion
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Spectral Norm Accelerator

$$\gamma_{S_k} = \frac{\|U_k^{-1}\|_{S}^{\frac{1}{2}}}{\|U_k\|_{S}^{\frac{1}{2}}}$$

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
			00	

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications ○●	Conclusion
Rotation Matrices				

What's Up with *U*?

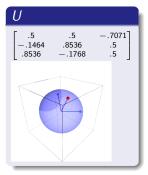
U =	$= R_{\theta} R_{\psi} R_{\kappa} V^{*}$			
		$\begin{array}{l} \cos\psi\sin\kappa\\ \sin\theta\sin\psi\sin\kappa+\cos\theta\cos\kappa\\ \cos\theta\sin\psi\sin\kappa-\sin\theta\cos\kappa\end{array}\end{array}$	$-\sin\psi\\\sin\theta\cos\psi\\\cos\theta\cos\psi\end{bmatrix}V^*$	

The 000	Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
Р	and <i>r</i>				
	r				
	$r = \sqrt{x^2 + y^2}$	2			

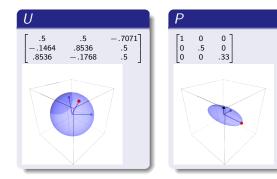
The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
P and r				
r				
$r = \sqrt{x^2 + y^2}$	2			
r Vector				
$r \text{ Vector}$ $\ \mathbf{r}\ = \sqrt{\mathbf{r}^* \mathbf{r}}$				

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
P and r				
r				
$r = \sqrt{x^2 + y^2}$	2			
r Vector				
$r \text{ Vector}$ $\ \mathbf{r}\ = \sqrt{\mathbf{r}^* \mathbf{r}}$				
P				
$P = \sqrt{A^*A}$				

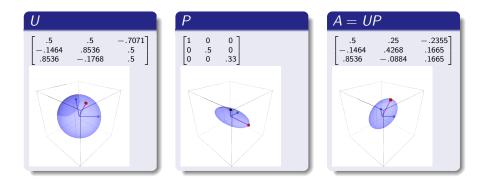
The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
Ideal Example				



The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications	Conclusion
Ideal Example				



The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Ideal Example				



The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Applications				

Use

Continuum Mechanics

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion
Applications				

Use

Continuum Mechanics

Another Use

Computer Graphics

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
Iterative Methods for <i>U</i>				

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$$U_{k+1} = \frac{1}{2}(U_k + U_k^{-t}), \quad U_0 = A$$

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Iterative Meth	hods for 11			

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Iterative Meth	ods for 11			

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The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications	Conclusion

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts	Applications 00	Conclusion
Conclusion				

The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
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The Polar Decomposition	SVD and Polar Decomposition	Geometric Concepts 00000	Applications 00	Conclusion
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