Tournament Matrices

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Tournaments

- Digraph that represents the outcome of a round-robin tournament

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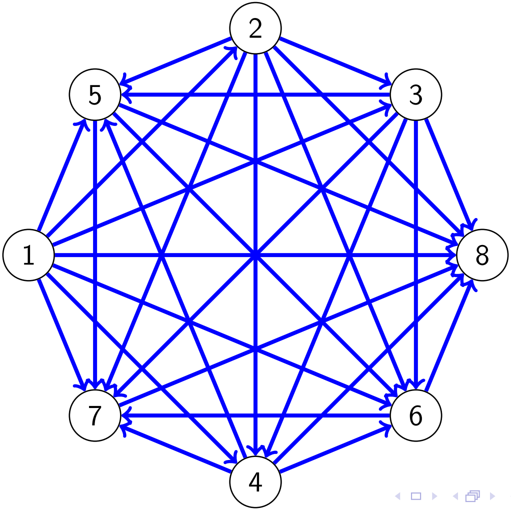
- Vertices are teams
- Edges denotes the victor between two teams
- K_n with direction

Example



Tournament Matrices

- Adjacency matrix of a tournament

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- Adjacency matrix of a tournament
- 1's represent wins, 0's represent losses

Properties

Let A be a tournament matrix of size $n \times n$

- $[A]_{ii} = 0$ for $1 \leq i \leq n$
- $[A]_{ij} + [A]_{ji} = 1$ for $1 \leq i < j \leq n$
- $A + A^T = J_n - I_n$
- $\sum_{i=1}^{n} \sum_{j=1}^{n} [A]_{ij} = \binom{n}{2}$

Row and Column Sums

- Row sum vector $R = (r_1, r_2, ..., r_n)$ where $r_i = \sum_{j=1}^{n} [A]_{ij}$
  - r_i represents the number of wins team i has
  - Also known as the score vector
  - $R = R(A) = Aj_n$

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  - $R = R(A) = Aj_n$
- Column sum vector $S = (s_1, s_2, ..., s_n)$ where $s_i = \sum_{i=1}^{n} [A]_{ij}$
  - s_i represents the number of losses team i has

Example

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Generalized Tournament Matrices

Generalized Tournament Matrices

- Tournament matrix where the values of the entries are 0 and 1 inclusive
- Entries are the probabilities that one team will defeat another

Regular Tournament Matrices

A tournament matrix A of size n with score vector R is a regular tournament matrix if

- n is odd
- Every entry of R is $(n-1)/2$

Example:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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- Normal ($AA^* = A^*A$)
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- Spectral radius $\rho = \rho(A) = (n-1)/2$
- $Aj_n = \rho j_n$
- Tournament matrices of size n where n is odd with the largest spectral radius are regular

Near-Regular Tournament Matrices

A tournament matrix A of size n with score vector R is a near-regular tournament matrix if

- n is even
- Half the entries of R are $(n-2)/2$ and the other half are $n/2$

Example: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Construction

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Theorem

Let A be any $n \times n$ tournament matrix. Then,

$$M_A = \begin{bmatrix} A & A^T \\ A^T + I_n & A \end{bmatrix}$$

is a $2n \times 2n$ near-regular tournament matrix.

Proof.
Since $A + A^T = J_n - I_n$, the first n rows of M_A have row sum $n-1$ and the last n rows of M_A have row sum n. So the score vector of M_A is

$$M_A j_{2n} = \begin{bmatrix} (n-1)j_n \\ nj_n \end{bmatrix}.$$

Therefore, by definition, M_A is a near-regular tournament matrix. □

Brualdi-Li Matrix

Near-regular tournament matrix of size $2m$ defined as

$$\mathcal{B}_{2m} = \begin{bmatrix} L_m & L_m^T \\ L_m^T + I_m & L_m \end{bmatrix}$$

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Properties:

- $\rho(\mathcal{B}_{2m}) \geq \rho(A)$ for every $2m \times 2m$ tournament matrix A
- If $\rho(\mathcal{B}_{2m}) = \rho(A)$, $PAP^T = \mathcal{B}_{2m}$ where P is some permutation matrix
- Diagonalizable, though not unitarily
- First m entries of score vector are $n - 1$. Last m entries are m.

Perron-Frobenius Theorem

Theorem

Let M be a nonnegative, irreducible matrix. Then the spectral radius of M, $\rho(M)$, is a unique, positive eigenvalue for M, and there is an entrywise positive eigenvector v. Such a vector v is called the Perron vector for ρ.

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- This is the sum of all entries in the i^{th} row of $A^2 \rightarrow A^2 j_n$ is the vector whose i^{th} entry is the sum of the scores of all teams defeated by team i

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- This is the sum of all entries in the i^{th} row of $A^2 \rightarrow A^2 j_n$ is the vector whose i^{th} entry is the sum of the scores of all teams defeated by team i
- Continue process up to $A^k j_n$ where k is an arbitrary positive integer pause
- $\lim_{k\rightarrow\infty} \frac{A^k j_n}{||A^k j_n||}$ =Perron vector v (Power Method)

Ramanujacharyula Ranking

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- Strength determined by right Perron vector v $(Av = \rho v)$
- Weakness determined by left Perron vector w $(w^T A = \rho w^T)$
- $w = \lim_{k\to\infty} \frac{j_n^T A^k}{||j_n^T A^k||}$

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- $w = \lim_{k\to\infty} \dfrac{j_n^T A^k}{||j_n^T A^k||}$
- Team i is stronger than team j if $v_i/w_i > v_j/w_j$.

Brualdi-Li Matrix and Rankings

Let $\mathcal{B}_{2m}$ be the Brualdi-Li matrix of size $2m$ with right Perron vector v and left Perron vector w

- Kendall-Wei Ranking:
  $v_{2m} < v_{2m-1} < v_{2m-2} < \ldots < v_{m+1} < v_1 < v_2 < \ldots < v_m$
- Ramanujacharyula Ranking:
  $\frac{v_m}{w_m} < \frac{v_1}{w_1} < \frac{v_{m-1}}{w_{m-1}} < \frac{v_2}{w_2} < \frac{v_{m-2}}{w_{m-2}} < \ldots < \frac{v_{m/2}}{w_{m/2}} < 1,$
  $1 < \frac{v_{2m-m/2+1}}{w_{2m-m/2+1}} < \ldots < \frac{v_{m+3}}{w_{m+3}} < \frac{v_{2m-1}}{w_{2m-1}} < \frac{v_{2m}}{w_{2m}} < \frac{v_{m+1}}{w_{m+1}}$
  where $m/2$ is rounded up if m is odd

Properties:

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- Both ranking schemes of $\mathcal{B}_{2m}$ agree with ranking via score vector
- Among all touraments with an even number of teams, the Brualdi-Li Matrix has minimal variation in rankings (well-matched teams)
- Left Perron vector and right Perron vector are transposes of eachother

Probabilities

Let v be the Perron vector of a tournament matrix A

- Probability team i beats team j is $\pi_{ij} = \dfrac{v_i}{v_i + v_j}$
- Generalized tournament matrix G: $[G]_{ij} = \pi_{ij}$

Big Example

Consider $\mathcal{B}_{12}$:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

The right Perron vector is

$$v = \lim_{k\to\infty} \frac{\mathcal{B}_{12}^k 1_{12}}{||\mathcal{B}_{12}^k 1_{12}||} =$$
$$\begin{bmatrix} .282 & .279 & .275 & .269 & .261 & .250 & .296 & .298 & .302 & .307 & .313 & .323 \end{bmatrix}$$

and the left Perron vector is

$$w = \lim_{k\to\infty} \frac{1_{12}^T \mathcal{B}_{12}^k}{||1_{12}^T \mathcal{B}_{12}^k||} =$$
$$\begin{bmatrix} .323 & .313 & .307 & .302 & .298 & .296 & .250 & .261 & .269 & .275 & .279 & .282 \end{bmatrix}$$

with decimals rounded to three significant figures.

Strength to weakness ratios

$$\frac{v_6}{w_6} = .845 < \frac{v_1}{w_1} = .873 < \frac{v_5}{w_5} = .876 < \frac{v_2}{w_2} = .890 < \frac{v_4}{w_4} = .891 < \frac{v_3}{w_3} = .896 < 1$$

$$1 < \frac{v_{10}}{w_{10}} = 1.116 < \frac{v_9}{w_9} = 1.122 < \frac{v_{11}}{w_{11}} = 1.123 < \frac{v_8}{w_8} = 1.142 < \frac{v_{12}}{w_{12}} = 1.145 < \frac{v_7}{w_7} = 1.184$$

Ranking according to Kendall-Wei:

$$12, 11, 10, 9, 8, 7, 1, 2, 3, 4, 5, 6$$

Ranking according to Ramanucharyula:

$$7, 12, 8, 11, 9, 10, 3, 4, 2, 5, 1, 6.$$

Generalized Tournament Matrix:

$$\begin{bmatrix} 0 & .503 & .506 & .512 & .519 & .530 & .488 & .486 & .483 & .479 & .474 & .466 \\ .497 & 0 & .504 & .509 & .517 & .527 & .485 & .484 & .480 & .476 & .471 & .463 \\ .494 & .496 & 0 & .506 & .513 & .524 & .482 & .480 & .477 & .473 & .468 & .460 \\ .488 & .491 & .494 & 0 & .507 & .518 & .476 & .474 & .471 & .467 & .462 & .454 \\ .481 & .483 & .487 & .493 & 0 & .511 & .469 & .467 & .464 & .460 & .455 & .447 \\ .470 & .473 & .476 & .482 & .489 & 0 & .458 & .456 & .453 & .449 & .444 & .436 \\ .512 & .515 & .518 & .524 & .531 & .542 & 0 & .498 & .495 & .491 & .486 & .478 \\ .514 & .516 & .520 & .526 & .533 & .544 & .502 & 0 & .497 & .493 & .488 & .480 \\ .517 & .520 & .523 & .529 & .536 & .547 & .505 & .503 & 0 & .496 & .491 & .483 \\ .521 & .524 & .527 & .533 & .540 & .551 & .509 & .507 & .504 & 0 & .495 & .487 \\ .526 & .529 & .532 & .538 & .545 & .556 & .514 & .512 & .509 & .505 & 0 & .492 \\ .534 & .537 & .540 & .546 & .553 & .564 & .522 & .520 & .517 & .513 & .508 & 0 \end{bmatrix}$$