Solving Toeplitz Systems of Equations and Matrix Conditioning

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1 Introduction

In Linear Algebra, there is a tendency to generalize all matrices to certain groups: Hermitian, Unitary, Nonsingular, Singular,etc., but not look at more specific classes of matrices. An important matrix found throughout mathematics and in real-world applications is the Toeplitz matrix. This paper will review a few specific ways of solving Toeplitz systems of equations using Block Gaussian Elimination. I will also address the importance of conditioning and its effect on Toeplitz matrices.

2 Toeplitz Matrices

A Toeplitz Matrix or Diagonal Constant Matrix is a nxn matrix where each of the descending diagonals are constant, where

$$T_n = \begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \ddots & t_{-2} \\ \vdots & \ddots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{bmatrix}$$

Another way of writing a Toeplitz matrix is

$$T = \sum_{k=1}^{n} a_{-k} F^k + \sum_{k=0}^{n} a_k B^k$$

Where B and F are backward shift and forward shift matrices such that

$$B = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

Some other common types of Toeplitz Matrices are Circulant Matrices and Hankel Matrices, both which hold the same basic properties of a Toeplitz Matrix but have different additonal properties. Hankel Matrices are matrices of the form

$$H_n = \begin{bmatrix} h_{-3} & h_{-2} & h_{-1} & h_0 \\ h_{-2} & h_{-1} & h_0 & h_1 \\ h_{-1} & h_0 & h_1 & h_2 \\ h_0 & h_1 & h_2 & h_3 \end{bmatrix}$$

Circulant Matrices are matrices of the form

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_1 \\ c_1 & c_0 & \ddots & c_2 \\ \vdots & \ddots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}$$

We will only consider Toeplitz Matrices and not the Hankel and Circulant Matrix.
Toeplitz Matrices are persymmetric, they are also bisymmetric and centrosymmetric if the matrix is a symmetric matrix. Toeplitz Matrices also commute aysmptotically, or diagonalize in the same basis as $n \to \infty$. Their eigenvectors are sines and cosines. Toeplitz Systems are also related to Fast Fourier Tranforms (FFT). Toeplitz matrices are seen through out engineering, mathematics and science when looking at images and signals processing, Fourier Transforms, Hilbert Spaces, and problems involving trigometric moments.

Definition 2.1. *Let A be an n x n matrix such that A is persymmetric if it is symmetric about its anti-diagonal*

Definition 2.2. *Let A be a n x n matrix such that A is centrosymmetric if it is symmetric about the center*

Definition 2.3. *Let A be a n x n matrix. A is bisymmetric if only if A is centrosymmetric and either symmetric or antisymmetric*

3 Conditioning of a Matrix

In Numerical Linear Algebra, conditioning of a matrix is key to stablably solving any system of equations. Conditioning dictates how stable a certain algorithm is for a certain system of equations. Let M be an nxn nonsingular matrix so that the conditioning number of any nonsingular matrix is,

$$\kappa(M) = \left\|M\right\| \left\|M^{-1}\right\|$$

The equality above states that the conditioning number is defined by the multiplication of two matrix norms, M and its inverse M^{-1}. There are three common ways to solve matrix norms, the 1-Norm, 2-Norm and the ∞-Norm.

Definition 3.1. *Let A be an m x n matrix. The 1-norm, $\left\|A\right\|_1$ is equal to the maximum column sum, or for $1 \leq j \leq n$ and a_j is the jth column of A*

$$\left\|A\right\|_1 = max_j \sum_{k=1}^{n} a_{k,j}$$

Definition 3.2. *Let A be an m x n matrix. The 2-norm,$\left\|A\right\|_2$ is equal to the maximum singular value of A or*

$$\left\|A\right\|_2 = max_i \delta$$

Definition 3.3. *Let A be an m x n matrix. The ∞-norm,$\left\|A\right\|_\infty$ is equal to the maximum row sum or, for $1 \leq i \leq m$ and m_i is the ith row of A*

$$\left\|M\right\|_\infty = max\ _i \sum_{k=1}^{m} m_{i,k}$$

M is well-conditioned if $\kappa(M)$ is small; Since $\kappa(M) \geq 1$, small or large is relative to 1. Since all norms on a finite-dimensional space are equivalent, then which ever matrix norm you choose small will stay small and large will remain large relative to 1. if M is singular, $\kappa(M) = \infty$, so $\kappa(M)$ is very large then M is near a singular matrix, and is ill-conditioned. Thus if M is ill-conditioned then M is nearly singular.
if $Mx = b$ and $\hat{M}x = \hat{b}$ and if $\left\|M - \hat{M}\right\| < 1$ then

$$\frac{\left\|x - \hat{x}\right\|}{\left\|x\right\|} \leq \left(\frac{\kappa(M)}{1 - \kappa(M)(\left\|M - \hat{M}\right\| / \left\|M\right\|}\right) \left\{\frac{\left\|M - \hat{M}\right\|}{\left\|M\right\|} + \frac{\left\|b - \hat{b}\right\|}{\left\|b\right\|}\right\}$$

if $Mx = b$ is the problem we want to solve and $\hat{M}x = \hat{b}$ is the problem we have, then the inequality generally states that the relative error of $\hat{x}$ is approximately bounded by the condition number of the matrix times the relative error in the matrix and the right hand sie. Depending on how accurate we are required to be will change what is an acceptably large condition number.

3.1 Example of Conditioning

Next to show conditioning and the condition number practice, consider the Toeplitz matrix,

$$A = \begin{bmatrix} -1 & -2 & 1 & 1 & -1 \\ -1 & -1 & -2 & 1 & 1 \\ -1 & -1 & -1 & -2 & 1 \\ 5 & -1 & -1 & -1 & -2 \\ 0 & 5 & -1 & -1 & -1 \end{bmatrix}$$

and its inverse,

$$A^{-1} = \begin{bmatrix} -\frac{83}{298} & -\frac{15}{298} & -\frac{45}{298} & \frac{31}{298} & -\frac{39}{298} \\ -\frac{41}{298} & -\frac{11}{298} & -\frac{33}{298} & -\frac{17}{298} & \frac{31}{298} \\ -\frac{27}{298} & -\frac{109}{298} & -\frac{29}{298} & -\frac{33}{298} & -\frac{45}{298} \\ -\frac{9}{298} & \frac{63}{298} & -\frac{109}{298} & -\frac{11}{298} & -\frac{15}{298} \\ -\frac{169}{298} & -\frac{9}{298} & -\frac{27}{298} & -\frac{41}{298} & -\frac{83}{298} \end{bmatrix}$$

the 1-Norm, 2-Norm, and the ∞-Norm of A are

$$\left\|M\right\|_1 = max_j \sum_{k=1}^{n} a_{k,j} = 2$$
$$\left\|M\right\|_2 = max_i \delta \approx 2.355$$
$$\left\|M\right\|_\infty = \max\ _i \sum_{k=1}^{m} m_{i,k} = 2$$

the the 1-Norm, 2-Norm, and the ∞-Norm of A^{-1} are

$$\left\|A^{-1}\right\|_1 = max_j \sum_{k=1}^{n} \left|a_{k,j}^{-1}\right| \approx 1.104$$
$$\left\|A^{-1}\right\|_2 = max_i \delta \approx .2031$$
$$\left\|A^{-1}\right\|_\infty = \max\ _i \sum_{k=1}^{m} \left|a_{i,k}^{-1}\right| \approx 1.104$$

Now knowing the norm of A and A^{-1}, we can compute its condition number,

$$\begin{aligned} \kappa(A)_1 &= \left\|A\right\|_1 \left\|A^{-1}\right\|_1 \approx (2)(1.104) \approx 2.208 \\ \kappa(A)_2 &= \left\|A\right\|_2 \left\|A^{-1}\right\|_2 \approx (2.2355)(.2031) \approx 0.4540 \\ \kappa(A)_\infty &= \left\|A\right\|_\infty \left\|A^{-1}\right\|_\infty \approx (2)(1.104) \approx 2.208 \end{aligned}$$

We can conclude that A is small relative to 1, therefore we can conclude that A is a well-conditioned matrix.

4 Block Gaussian Elimination

We partition the system $Tx = b$, where T is a nonsingular and symmetric Toeplitz matrix, into small systems

$$Mx = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x} \\ \breve{x} \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \breve{b} \end{bmatrix} = b$$

where x and b are nx1, A is (kxk), B is kx$(n-k)$, C is $(n-k)$xk, D is $(n-k)$x$(n-k)$, $\hat{x}$ and $\hat{b}$ are kx1 and $\breve{x}$ and $\breve{b}$ are $(n-k)$x1.
Using block Guassian elimination and asssuming A is nonsingular,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & \Delta \end{bmatrix}$$

Where $\Delta = D - CA^{-1}B$, and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x} \\ \breve{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{b} \\ \breve{b} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \breve{x} - CA^{-1}\hat{x} \end{bmatrix}$$

So we solve $Tx = b$ by the following:

1. Solving $AX = C$ for X, where X is $(n-k)$xk matrix
2. Forming $\Delta = D - XB$
3. Forming $\breve{c} = \breve{b} - X\hat{b}$
4. Solving $\Delta\breve{x} = \breve{c}$ for $\breve{x}$
5. Forming $\hat{c} = \hat{b} - B\hat{x}$ and
6. Solving $A\hat{x} = \hat{c}$ for $\hat{x}$.

If M is nonsingular then A does not have to be nonsingular. Since M and A are nonsingular, Δ is guarenteed to be nonsingular since $\det(M) = \det(A)\det(\Delta)$.

4.1 Conditioning and Block Guassian Elimination

Though M might be well-conditioned, this does not produce instant stability for the algorithm. If A is also well-conditioned then the algorithm will be generally stable. The only class of matrices that would produce a well-conditioned A is a symmetric positive definite matrices: $x^T M x > 0$ for all $x \leq 0$ or equivalently, $M = M^T$ and all eigenvalues of M are positive. Let

$$M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

be symmetric positive definite, thus A, D and $\Delta = D - B^T A^{-1} B$ are symmetric positive definite. In this specific case we shall use the 2-norm, since all norms are equivalent in finite dimensional spaces,

$$\kappa_2(M) = \frac{\sigma_{max}(M)}{\sigma_{min}(M)} \qquad \kappa_2(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

where σ_{max} is the largest singular value and σ_{min} is the smallest. Since M and A are symmetric positive definite, $\sigma_{max}(M) = \lambda_{max}(M)$, $\sigma_{min}(M) = \lambda_{min}(M)$, $\sigma_{max}(A) = \lambda_{max}(A)$, $\sigma_{min}(A) = \lambda_{min}(A)$, where λ_{max} is the largest eigenvalue and λ_{min} is the smallest. By Cauchy Interlace Theorem, Let Let A be a symmetric n x n matrix. The $m \quad m$ matrix B, where $mleqn$ is called a compression of A if there exists an orthogonal projection P onto a subspace of dimension m such that $P^*AP = B$.

Theorem 4.1. *If the eigenvalues of A are $\alpha_1 \leq \cdots \leq \alpha_n$, and those of B are $\beta_1 \leq \cdots \leq \beta_j \leq \cdots \leq \beta_m$ then for all $j < m+1$*

$$\alpha_j \leq \beta_j \leq \alpha_{n-m+j}$$

Thus,

$$\kappa_2(A) = \frac{\sigma max(A)}{\sigma min(A)} = \frac{\lambda max(A)}{\lambda min(A)} \leq \frac{\lambda max(M)}{\lambda min(M)} = \frac{\sigma max(M)}{\sigma min(M)} = \kappa_2(M)$$

Then, if M is well conditioned then A is also well-conditioned, or equivalently A is ill conditioned then M is ill-conditioned.

5 Large Example of Block Gaussian Elimination and Conditioning

We will conclude with one overarching example of both block Gaussian elimination and conditioning. Consider the matrix T

$$T = \begin{bmatrix} 1 & 2 & 0 & -1 & 5 & 8 \\ 2 & 1 & 2 & 0 & -1 & 5 \\ 0 & 2 & 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 & 2 & 0 \\ 5 & -1 & 0 & 2 & 1 & 2 \\ 8 & 5 & -1 & 0 & 2 & 1 \end{bmatrix}$$

where T is also symmetric, nonsingular and positive definite.
Before we start partitioning for block Gaussian elimination, we must first check the condition number to see how well or ill-conditioned T actually is.

$$\begin{aligned} \|T\|_1 &= 15 \\ \|T\|_2 &\approx 12.822 \\ \|T\|_\infty &= 15 \end{aligned}$$

Now we find the matrix norms of T^{-1},

$$\begin{aligned} \|T^{-1}\|_1 &\approx .284 \\ \|T^{-1}\|_2 &\approx .784 \\ \|T^{-1}\|_\infty &\approx .284 \end{aligned}$$

After computing the 1-norm, 2-norm and ∞-norm of T, we now compute the conditioning number,

$$\begin{aligned}\kappa(T)_1 &= \left\|T\right\|_1 \left\|T^{-1}\right\|_1 = (15)(.284) = 4.26\\ \kappa(T)_2 &= \left\|T\right\|_2 \left\|T^{-1}\right\|_2 = (12.822)(.784) = 10.05\\ \kappa(T)_\infty &= \left\|T\right\|_\infty \left\|T^{-1}\right\|_\infty = (15)(.284) = 4.26\end{aligned}$$

since $\kappa(A)$ is small relative to 1, we can say that T is well-conditioned.
To be safe, we check A to prove that A is also well-conditioned. We will use the 2-norm method for A,

$$\begin{aligned}\left\|A\right\| &\approx 3.828\\ \left\|A^{-1}\right\| &= 1\end{aligned}$$

We now compute $\kappa(A)_2$,

$$\kappa(A)_2 = \frac{\delta_{max}}{\delta_{min}} = \frac{\lambda_{max}}{\lambda_{min}} \approx 3.828$$

This produces the inequality proved in Section 4.1 needed to say A is also well conditioned

$$\begin{aligned}\kappa(A)_2 &\leq \kappa(M)_2\\ 3.828 &\leq 10.05\end{aligned}$$

Let us now consider the system $Tx = b$. Using block Gaussian elimination we partition the system such that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 5 & 8 \\ 0 & -1 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 2 \\ 5 & -1 & 0 \\ 8 & 5 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; \breve{x} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \; \hat{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \; \breve{b} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

where

$$CA^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 5 & -1 & 0 \\ 8 & 5 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & -\frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix} = \begin{bmatrix} -\frac{11}{7} & \frac{2}{7} & \frac{10}{7} \\ \frac{13}{7} & \frac{11}{7} & -\frac{22}{7} \\ \frac{38}{7} & \frac{9}{7} & -\frac{25}{7} \end{bmatrix}$$

$$\Delta = D - CA^{-1}B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -\frac{11}{7} & \frac{2}{7} & \frac{10}{7} \\ \frac{13}{7} & \frac{11}{7} & -\frac{22}{7} \\ \frac{38}{7} & \frac{9}{7} & -\frac{25}{7} \end{bmatrix} \begin{bmatrix} -1 & 5 & 8 \\ 0 & -1 & 5 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{24}{7} & \frac{71}{7} & \frac{88}{7} \\ \frac{71}{7} & -\frac{47}{7} & -\frac{167}{7} \\ \frac{88}{7} & -\frac{167}{7} & -\frac{367}{7} \end{bmatrix}$$

knowing CA^{-1} and Δ, we can successfully decompose T,

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & \Delta \end{bmatrix}$$

where

$$\begin{bmatrix} A & B \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \hat{x} \\ \breve{x} \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \breve{b} - CA^{-1}\hat{b} \end{bmatrix}$$

Now to finally solve for x, let

$$\breve{c} = \breve{b} - CA^{-1}\hat{b} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{11}{7} & \frac{2}{7} & \frac{10}{7} \\ \frac{13}{7} & \frac{11}{7} & -\frac{22}{7} \\ \frac{38}{7} & \frac{9}{7} & -\frac{25}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{19}{7} \\ -\frac{67}{7} \\ -\frac{65}{7} \end{bmatrix}$$

then solving for $\breve{x}$

$$\breve{x} = \Delta^{-1}\breve{c} = \begin{bmatrix} -\frac{1520}{7807} & \frac{1623}{7807} & -\frac{1103}{7807} \\ \frac{1623}{7807} & \frac{152}{7807} & \frac{320}{7807} \\ -\frac{1103}{7807} & \frac{320}{7807} & -\frac{559}{7807} \end{bmatrix} \begin{bmatrix} \frac{19}{7} \\ -\frac{67}{7} \\ -\frac{65}{7} \end{bmatrix} = \begin{bmatrix} -\frac{9418}{7807} \\ -\frac{21}{7807} \\ -\frac{866}{7807} \end{bmatrix} \approx \begin{bmatrix} -1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix}$$

Finally knowing $\breve{x}$, we can finally solve for $\hat{x}$.

$A\hat{x} = \hat{b} - B\breve{x}$

$$\hat{x} = A^{-1}(\hat{b} - B\breve{x})\hat{x} \quad = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & -\frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} & \frac{2}{7} \\ -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix} \left[\frac{5422}{7807}, \frac{12116}{7807}, \frac{10163}{7807}\right] = \begin{bmatrix} -\frac{22}{7807} \\ \frac{2722}{7807} \\ \frac{4719}{7807} \end{bmatrix} \approx \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \end{bmatrix}$$

Therefore

$$x = \begin{bmatrix} \hat{x} \\ \breve{x} \end{bmatrix} = \begin{bmatrix} -0.00281798386064 \\ 0.348661457666 \\ 0.604457538107 \\ 1.2063532727 \\ -0.00268989368515 \\ -0.110926091969 \end{bmatrix}$$

The beauty of block Gaussian elimination to solve Toeplitz systems of equations is that each partition maintains Toeplitz structure. Each partition remains bisymmetric and nonsingular. Block Gaussian elimination only requires $O(n^2)$ flops which is a relatively good speed for an algorithm, but in modern day mathematics and computer science, the fascination with these new aysmptotically fast algorithms may have the block Gaussian elimination become less relevent.

6 Conclusion

Toeplitz systems of equations are a very unique type of systems that can be ,at times, difficult to solve. Mathematicians and Computer Scientists have been forging different ways to effectively and quickly solve these systems. Like any matrix, Toeplitz matrices are severely effected by conditioning, and how that effects a specifics algorithms overall stability. The added benefit that Toeplitz matrices have are that in certain situations, the positive defintie and Hermitian case, can nicely be decomposed, partitioned or factored.

7 Bibliography

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