Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2015
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of ~~100~~ 90 possible points.
You may use Sage to manipulate and row-reduce matrices. Be sure to make it clear what you have input to Sage, and show any output you use to justify your answers.

1. Does the set S span the vector space of 2×3 matrices, M_{23}? (10 points)

$$S = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -5 & -6 \\ 2 & -1 & -6 \end{bmatrix}, \begin{bmatrix} -3 & -4 & -4 \\ 4 & -3 & -7 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -5 \\ -8 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 & 8 \\ 5 & -5 & 0 \end{bmatrix} \right\}$$

$\dim(M_{23}) = 2\cdot 3 = 6$ by Theorem DM.

size of $S = 5 < 6 = \dim(M_{23})$ so by Theorem G, S does not span M_{23}

2. Is the set T linearly independent in the vector space of polynomials with degree 2 or less, P_2? (10 points)

$T = \{x^2+3x+3,\ 2x^2+7x+6,\ 2x^2+4x+7\}$

$\dim(P_2) = 3$ by Theorem DP, so T could span P_2. Check with a relation of linear dependence.

$$\mathbf{0} = a_1(x^2+3x+3) + a_2(2x^2+7x+6) + a_3(2x^2+4x+7)$$

$$0x^2+0x+0 = (a_1+2a_2+2a_3)x^2 + (3a_1+7a_2+4a_3)x + (3a_1+6a_2+7a_3)$$

leads to homogeneous system w/ coefficient matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 4 \\ 3 & 6 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $a_1 = a_2 = a_3 = 0$ & T is linearly independent in P_2

3. Is the set R a basis of the vector space $\mathbb{C}^4$? (10 points)

$$R = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ -2 \\ -5 \end{bmatrix} \right\} = \{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4\};\quad A = [\mathbf{R}_1|\mathbf{R}_2|\mathbf{R}_3|\mathbf{R}_4]$$

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem NMRRI, A is nonsingular. By Theorem NME6, the columns of A (aka R) is a basis of $\mathbb{C}^4$.

4. Prove that the set $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \, 3a + 5b = 0 \right\}$ is a subspace of the vector space of column vectors $\mathbb{C}^2$. (15 points)

(a) In $\mathbb{C}^2$, $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $3(0)+5(0)=0$. So $\underset{\sim}{0} \in W$.

(b) Suppose $\underset{\sim}{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \underset{\sim}{y} = \begin{bmatrix} c \\ d \end{bmatrix} \in W$. Then $3a+5b=0$, $3c+5d=0$.

We have $\underset{\sim}{x}+\underset{\sim}{y} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$. Consider then,

$3(a+c)+5(b+d) = 3a+3c+5b+5d = (3a+5b)+(3c+5d) = 0+0 = 0$.

Thus, $\underset{\sim}{x}+\underset{\sim}{y} \in W$.

(c) Suppose $\alpha \in \mathbb{C}$, $\underset{\sim}{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in W$. Then $3a+5b=0$.

We have $\alpha \underset{\sim}{x} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$. Consider then,

$3(\alpha a)+5(\alpha b) = \alpha(3a+5b) = \alpha \cdot 0 = 0$. So $\alpha \underset{\sim}{x} \in W$.

So by Theorem TSS, W is a subspace of $\mathbb{C}^2$

5. The set $W = \{ a + bx + cx^2 \,|\, a + 2b - 3c = 0 \}$ is a subspace of the vector space of polynomials in x with degree 2 or less, P_2. (You may assume this.) Determine, with verification, a basis of W. (20 points)

$a+2b-3c=0 \rightarrow a=-2b+3c$.

$W = \{(-2b+3c)+bx+cx^2 \mid b,c \in \mathbb{C}\}$

$= \{(-2b+bx)+(3c+cx^2) \mid b,c \in \mathbb{C}\}$

$= \{b(-2+x)+c(3+x^2) \mid b,c \in \mathbb{C}\} = \langle \{-2+x, 3+x^2\} \rangle$

So $B = \{-2+x, 3+x^2\}$ <u>is</u> a spanning set for W.

Is B linearly independent?

$\underset{\sim}{0} = \alpha_1(-2+x)+\alpha_2(3+x^2)$

$0x^2+0x+0 = (-2\alpha_1+3\alpha_2)+\alpha_1 x+\alpha_2 x^2$

$\Rightarrow \underbrace{-2\alpha_1+3\alpha_2=0, \quad \alpha_1=0, \quad \alpha_2=0}$

implies B <u>is</u> linearly independent

So B <u>is</u> a basis of W.

6. Suppose that the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent subset of the vector space V. Prove that the set $T = \{3\mathbf{u} + 4\mathbf{v} - 8\mathbf{w}, -\mathbf{u} - \mathbf{v} + 2\mathbf{w}, 2\mathbf{u} + 2\mathbf{v} - 3\mathbf{w}\}$ is linearly independent in V. (15 points)

RLD $\quad \mathbf{0} = a_1(3\mathbf{u} + 4\mathbf{v} - 8\mathbf{w}) + a_2(-\mathbf{u} - \mathbf{v} + 2\mathbf{w}) + a_3(2\mathbf{u} + 2\mathbf{v} - 3\mathbf{w})$

$= (3a_1 - a_2 + 2a_3)\mathbf{u} + (4a_1 - a_2 + 2a_3)\mathbf{v} + (-8a_1 + 2a_2 - 3a_3)\mathbf{w}$

Several vector space properties allow us to get here.

Now have a RLD on the linearly independent set S.

So
$$\begin{aligned} 0 &= 3a_1 - a_2 + 2a_3 \\ 0 &= 4a_1 - a_2 + 2a_3 \\ 0 &= -8a_1 + 2a_2 - 3a_3 \end{aligned} \quad \xRightarrow[\text{techniques}]{\text{usual}} \quad a_1 = a_2 = a_3 = 0 \text{ is the only solution.}$$

Thus T is linearly independent.

7. Suppose that $R = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ spans the vector space $\mathbb{C}^n$ and that A is an $n \times n$ nonsingular matrix. Prove that $P = \{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_m\}$ spans $\mathbb{C}^n$. (10 points)

① Grab arbitrary $\mathbf{w} \in \mathbb{C}^n$.

② Solve $\mathcal{LS}(A, \mathbf{w})$. Always possible because A is nonsingular. Call a solution $\mathbf{u}$, so $A\mathbf{u} = \mathbf{w}$.

③ R spans, so there are scalars $a_1, a_2, \ldots, a_m$ so that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = \mathbf{u}$.

④ Now
$$\begin{aligned} \mathbf{w} &= A\mathbf{u} \\ &= A(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m) \\ &= a_1 A\mathbf{v}_1 + a_2 A\mathbf{v}_2 + \cdots + a_m A\mathbf{v}_m \end{aligned}$$

So any $\mathbf{w}$ is a linear combination of elements of P. So P spans $\mathbb{C}^n$.