

Show *all* of your work and *explain* your answers fully. There is a total of ~~100~~ possible points.
You may use Sage to manipulate and row-reduce matrices. Be sure to make it clear what you have input to Sage, and show any output you use to justify your answers.

1. Does the set S span the vector space of 2×3 matrices, M_{23} ? (10 points)

$$S = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -5 & -6 \\ 2 & -1 & -6 \end{bmatrix}, \begin{bmatrix} -3 & -4 & -4 \\ 4 & -3 & -7 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -5 \\ -8 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 & 8 \\ 5 & -5 & 0 \end{bmatrix} \right\}$$

$\dim(M_{23}) = 2 \cdot 3 = 6$ by Theorem DM.

size of $S = 5 < 6 = \dim(M_{23})$ so by Theorem G, S does not span M_{23}

2. Is the set T linearly independent in the vector space of polynomials with degree 2 or less, P_2 ? (10 points)

$T = \{x^2 + 3x + 3, 2x^2 + 7x + 6, 2x^2 + 4x + 7\}$ $\dim(P_2) = 3$ by Theorem DP, so T could span P_2 . Check with a relation of linear dependence.

$$0 = a_1(x^2 + 3x + 3) + a_2(2x^2 + 7x + 6) + a_3(2x^2 + 4x + 7)$$

$$0x^2 + 0x + 0 = (a_1 + 2a_2 + 2a_3)x^2 + (3a_1 + 7a_2 + 4a_3)x + (3a_1 + 6a_2 + 7a_3)$$

leads to homogeneous system of coefficient matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 4 \\ 3 & 6 & 7 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

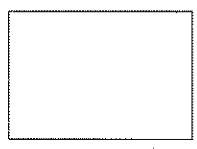
So $a_1 = a_2 = a_3 = 0$
 $\neq T$ is linearly independent in P_2

3. Is the set R a basis of the vector space \mathbb{C}^4 ? (10 points)

$$R = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ -2 \\ -5 \end{bmatrix} \right\} = \{ \underline{R}_1, \underline{R}_2, \underline{R}_3, \underline{R}_4 \}; A = [\underline{R}_1 | \underline{R}_2 | \underline{R}_3 | \underline{R}_4]$$

$$A \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By Theorem NMRRI, A is nonsingular.
By Theorem NME6, the columns of A (aka R) is a basis of \mathbb{C}^4 .



4. Prove that the set $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid 3a + 5b = 0 \right\}$ is a subspace of the vector space of column vectors \mathbb{C}^2 . (15 points)

(a) In \mathbb{C}^2 , $\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $3(0) + 5(0) = 0$. So $\underline{0} \in W$.

(b) Suppose $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\underline{y} = \begin{bmatrix} c \\ d \end{bmatrix} \in W$. Then $3a + 5b = 0$
 $3c + 5d = 0$.

We have $\underline{x} + \underline{y} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$. Consider then,
 $3(a+c) + 5(b+d) = 3a + 3c + 5b + 5d = (3a + 5b) + (3c + 5d) = 0 + 0 = 0$.

Thus, $\underline{x} + \underline{y} \in W$.

(c) Suppose $\alpha \in \mathbb{C}$, $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in W$. Then $3a + 5b = 0$.

We have $\alpha \underline{x} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$. Consider then,

$3(\alpha a) + 5(\alpha b) = \alpha(3a + 5b) = \alpha \cdot 0 = 0$. So $\alpha \underline{x} \in W$.

So by Theorem TSS, W is a subspace of \mathbb{C}^2

5. The set $W = \{a + bx + cx^2 \mid a + 2b - 3c = 0\}$ is a subspace of the vector space of polynomials in x with degree 2 or less, P_2 . (You may assume this.) Determine, with verification, a basis of W . (20 points)

$$a + 2b - 3c = 0 \rightarrow a = -2b + 3c.$$

$$\begin{aligned} W &= \{(-2b + 3c) + bx + cx^2 \mid b, c \in \mathbb{C}\} \\ &= \{(-2b + bx) + (3c + cx^2) \mid b, c \in \mathbb{C}\} \\ &= \{b(-2 + x) + c(3 + x^2) \mid b, c \in \mathbb{C}\} = \langle \{-2 + x, 3 + x^2\} \rangle \end{aligned}$$

So $B = \{-2 + x, 3 + x^2\}$ is a spanning set for W .

Is B linearly independent?

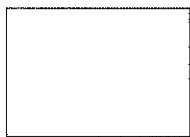
$$\underline{0} = \alpha_1(-2 + x) + \alpha_2(3 + x^2)$$

$$0x^2 + 0x + 0 = (-2\alpha_1 + 3\alpha_2) + \alpha_1x + \alpha_2x^2$$

$$\Rightarrow -2\alpha_1 + 3\alpha_2 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = 0$$

implies B is linearly
² independent

So B is a
 basis of W .



6. Suppose that the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent subset of the vector space V . Prove that the set $T = \{3\mathbf{u} + 4\mathbf{v} - 8\mathbf{w}, -\mathbf{u} - \mathbf{v} + 2\mathbf{w}, 2\mathbf{u} + 2\mathbf{v} - 3\mathbf{w}\}$ is linearly independent in V . (15 points)

RLD
$$\underline{0} = a_1(3\underline{u} + 4\underline{v} - 8\underline{w}) + a_2(-\underline{u} - \underline{v} + 2\underline{w}) + a_3(2\underline{u} + 2\underline{v} - 3\underline{w})$$

$$= (3a_1 - a_2 + 2a_3)\underline{u} + (4a_1 - a_2 + 2a_3)\underline{v} + (-8a_1 + 2a_2 - 3a_3)\underline{w}$$

Several vector space properties allow us to get here.
Now have a RLD on the linearly independent set S .

So
$$\begin{aligned} 0 &= 3a_1 - a_2 + 2a_3 && \text{usual} && a_1 = a_2 = a_3 = 0 \\ 0 &= 4a_1 - a_2 + 2a_3 && \implies && \text{is the only} \\ 0 &= -8a_1 + 2a_2 - 3a_3 && \text{techniques} && \text{solution.} \end{aligned}$$

Thus T is linearly independent.

7. Suppose that $R = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ spans the vector space \mathbb{C}^n and that A is an $n \times n$ nonsingular matrix. Prove that $P = \{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_m\}$ spans \mathbb{C}^n . (10 points)

① Grab arbitrary $\underline{w} \in \mathbb{C}^n$.

② Solve $LS(A, \underline{w})$. Always possible because A is nonsingular.
Call a solution \underline{u} , so $A\underline{u} = \underline{w}$.

③ R spans, so there are scalars a_1, a_2, \dots, a_m so that
$$a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_m \underline{v}_m = \underline{u}.$$

④ Now
$$\begin{aligned} \underline{w} &= A\underline{u} \\ &= A(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_m \underline{v}_m) \\ &= a_1 A\underline{v}_1 + a_2 A\underline{v}_2 + \dots + a_m A\underline{v}_m \end{aligned}$$

So any \underline{w} is a linear combination of elements of P . So P spans \mathbb{C}^n .