Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2015
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.
You may use Sage to manipulate matrices and vectors, and compute reduced row-echelon form, inverses, determinants and eigen-stuff. Be sure to make it clear what you have input to Sage, and show any output you use to justify your answers. $\mathbb{C}^n$ is the vector space of column vectors with n entries, P_n is the vector space of polynomials with degree at most n and M_{mn} is the vector space of $m \times n$ matrices.

1. Compute the matrix representation of T relative to the bases B and C, $M^T_{B,C}$. (15 points)

$T\colon P_1 \to M_{12}, \qquad T(a+bx) = \begin{bmatrix} 2a+b & a-b \end{bmatrix}$

$B = \{1+2x, 3-x\} \qquad C = \left\{\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 5 \end{bmatrix}\right\}$

$\rho_C(T(1+2x)) = \rho_C([4\ \ -1]) = \rho_C(-23[1\ \ 2] + 9[3\ \ 5]) = \begin{bmatrix} -23 \\ 9 \end{bmatrix}$

$\rho_C(T(3-x)) = \rho_C([5\ \ 4]) = \rho_C(-13[1\ \ 2] + 6[3\ \ 5]) = \begin{bmatrix} -13 \\ 6 \end{bmatrix}$

So $M^T_{B,C} = \begin{bmatrix} -23 & -13 \\ 9 & 6 \end{bmatrix}$

2. Use vector representations to efficiently answer the following questions. (15 points)

(a) Is $S = \{1 - 4x + 8x^2, 1 - 3x + 6x^2, -1 + 4x - 7x^2\}$ a linearly independent set in P_2?

Basis of P_2: $\{1, x, x^2\}$

Is $\left\{\begin{bmatrix} 1 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}\right\}$ linearly independent in $\mathbb{C}^3$?

$\begin{bmatrix} 1 & 1 & -1 \\ -4 & -3 & 4 \\ 8 & 6 & -7 \end{bmatrix} \xrightarrow{\text{RREF}} I_3$ So by Theorem LIVRN these are linearly independent sets.

(b) Does the set $Q = \{-7 - 3x + x^2, -5 - 2x + x^2, -3 - x + x^2\}$ span P_2?

Coordinatize again

$\left\{\begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}\right\}$ $\begin{bmatrix} -7 & -5 & -3 \\ -3 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\xrightarrow{\text{Theorem BS}}$ dimension 2

So Q only spans a subspace of P_2 with dimension 2.

3. Use a matrix representation for the following questions about the linear transformation T. (30 points)

$$T\colon M_{22} \to P_2, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (-7a - 5b + 10c - 31d) + (-2a - b + 2c - 8d)\,x + (-3a - 2b + 4c - 13d)\,x^2$$

(a) Compute the kernel of T, $\mathcal{K}(T)$.

Standard bases $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $C = \{1, x, x^2\}$

matrix representation $M^T_{B,C} = \begin{bmatrix} -7 & -5 & 10 & -31 \\ -2 & -1 & 2 & -8 \\ -3 & -2 & 4 & -13 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$N(M) = \left\langle \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

un coordinatize relative to B

$K(T) = \left\langle \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & -2 \\ 0 & 1 \end{bmatrix} \right\} \right\rangle$

(b) Based on your answer to the previous question, is T injective?

No $K(T) \neq \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(c) Find two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $T(\mathbf{x}) = T(\mathbf{y})$.

$T\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right) = 0 + 0x + 0x^2 = T\left(\begin{bmatrix} -3 & -2 \\ 0 & 1 \end{bmatrix} \right)$

(d) Compute the dimension of the range of T, $\dim(\mathcal{R}(T))$.

$\dim(R(T)) + \dim(K(T)) = \dim(M_{22})$

$r(T) + 2 = 4 \Rightarrow r(T) = 2$

(e) Based on your answer to the previous question, is T surjective?

By RoSLT, with $r(T) = 2 \neq 3 = \dim P_2$, no.

(f) Find a vector $\mathbf{x}$ whose preimage, $T^{-1}(\mathbf{x})$, is empty.

$(M^T_{B,C})^T \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in C(M^T_{B,C})$

& $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \notin C(M^T_{B,C})$

un coordinatize $\Rightarrow$

$T^{-1}(1+2x) = \emptyset$

4. Determine a basis B for P_2 so that the matrix representation of S relative to B is a diagonal matrix. (15 points)

$$S\colon P_2 \to P_2, \qquad S\left(a+bx+cx^2\right) = (-23a+12b+6c) + (-48a+25b+12c)\,x + (12a-6b-2c)\,x^2$$

Basis: $B = \{1, x, x^2\}$ Matrix rep

$$M^S_{B,B} = \begin{bmatrix} -23 & 12 & 6 \\ -48 & 25 & 12 \\ 12 & -6 & -2 \end{bmatrix}$$

• eigenmatrix_right()

$$\begin{bmatrix} -2 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1/2 & 4 & -2 \end{bmatrix} \qquad \left\{1+2x-\tfrac{1}{2}x^2,\ 1+4x^2,\ x-2x^2\right\}$$

un coordinatize eigenvectors (columns) to get desired basis

5. Compute an explicit formula for L^{-1}. (You may assume L is invertible.) (15 points)

$$L\colon \mathbb{C}^3 \to P_2, \qquad L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (3a-b+3c) + (4a-b+3c)\,x + (4a+2b-7c)\,x^2$$

Bases: $B = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ $\quad C = \{1, x, x^2\}$

$$M^T_{B,C} = \begin{bmatrix} 3 & -1 & 3 \\ 4 & -1 & 3 \\ 4 & 2 & -7 \end{bmatrix} \text{ then } M^{T^{-1}}_{C,B} = \left(M^T_{B,C}\right)^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -40 & 33 & -3 \\ -12 & 10 & -1 \end{bmatrix}$$

$$L^{-1}(a+bx+cx^2) = \rho_B^{-1}\left(M^{T^{-1}}_{C,B}\,\rho_C(a+bx+cx^2)\right) = \rho_B^{-1}\left(\begin{bmatrix} -1 & 1 & 0 \\ -40 & 33 & -3 \\ -12 & 10 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

$$= \rho_B^{-1}\left(\begin{bmatrix} -a+b \\ -40a+33b-3b \\ -12a+10b-c \end{bmatrix}\right) = \begin{bmatrix} -a+b \\ -40a+33b-3b \\ -12a+10b-c \end{bmatrix}$$

linear combination of B.