Formal Power Series

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Lucky tickets example

Lucky tickets of six digits

A ticket of six digits is a string, six numbers long, where the digits can take on values from 0 to 9. Then a "lucky ticket" is a ticket where the sum of the first three digits is equal to the sum of the last three digits (e.g., 030111 or 225900). The problem is to find how many lucky tickets there are in all the tickets of six digits.

Getting Started

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- What are possible values that the sum of three digits can take on? Any number from 0 to 27.
- For each of the above values, how many one-digit numbers are there whose digits sum up to that value?

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Getting Started

- What are possible values that the sum of three digits can take on? Any number from 0 to 27.
- For each of the above values, how many one-digit numbers are there whose digits sum up to that value? One, if the value is between 0 and 9. None, if the value is 10 or above.

Generating functions and formal power series

Formal Power Series

Let $\{a_n\} = a_0, a_1, a_2, \ldots$ be a sequence of numbers. Then the *formal power series* associated with $\{a_n\}$ is the series

$$A(s) = a_0 + a_1 s + a_2 s^2 + \ldots,$$

where s is a formal variable.

In the context of combinatorics, $A(s)$ is the *generating function* of $\{a_n\}$.

Generating functions in the lucky tickets problem

The series

$$A_1(s) = 1 + s + s^2 + s^3 + \ldots + s^9.$$

is the generating function where the coefficient a_n is the number of one-digit numbers whose digits add up to n.
What about the series where a_n is the number of two-digit numbers with sums of the digits equal to n?

- $a_0 = |\{00\}| = 1$
- $a_1 = |\{01, 10\}| = 2$
- $a_2 = |\{02, 11, 20\}| = 3$
- $\ldots$

So $A_2(s) = 1 + 2s + 3s^2 + \ldots$.

Take the square of $A_1(s)$.

$$(A_1(s))^2 = \sum_{i=0}^{9} \sum_{k=0}^{i} a_k a_{i-k} s^i.$$

Thus the coefficient of s^n is the sum of all $a_j a_{n-j}$, where $0 \leq j \leq n$. Each term $a_j a_{n-j}$ represents the total number of ways to write a two-digit number where the first digit is j and the second digit is $n-j$. This is the same as finding the number of two-digit numbers whose digits sum to n.

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Solving with generating functions

For each value n between 0 and 27, the coefficient a'_n of s^n in $(A_1(s))^3$ is equal to the ways that we can write the first three digits of a ticket and have the sum of those digits equal n. Then a'^2_n is the number of ways to write a lucky ticket where the sum of the first three digits is n and the sum of the last three digits is n.
To find the solution to the lucky tickets problem, multiply out

$$(1 + s + s^2 + \ldots + s^9)^3$$

and take the sum of the squares of each coefficient.

Generalizing

Notice that

- The variable s was not used as a function variable that we could substitute a value in for, but was useful for keeping a tally of powers
- We only looked at series with a finite number of nonzero terms; what if there are infinitely many?

Since it may be useful for other combinatorial applications to have a series with infinitely many or an unknown number of nonzero terms, we formalize the addition and multiplication of formal power series.

The ring of formal power series

Let R be a ring. We define $R[[s]]$ to be the set of formal power series in s over R. Then $R[[s]]$ is itself a ring.
Let $A(s) = a_0 + a_1 s + a_2 s^2 + \dots$ and $B(s) = b_0 + b_1 s + b_1 s^2 + \dots$ be elements of $R[[s]]$. Then

- the **sum** $A(s) + B(s)$ is defined to be $C(s) = c_0 + c_1 s + c_2 s^2 + \dots$, where $c_i = a_i + bi$ for all $i \geq 0$, and
- the **product** $A(s)B(s)$ is defined to be $C(s) = a_0 b_0 + (a_0 b_1 + a_1 b_0)s + \dots + \left(\sum_{i=0}^{k} a_i b_{k-i}\right) s^k + \dots$.

A few algebraic properties

- If R is an integral domain, then $R[[s]]$ is an integral domain.
- If R is a Noetherian ring, then $R[[s]]$ is a Noetherian ring.
- If R is a field, then $R[[s]]$ is a Euclidean domain.
- Let R be an integral domain. We can find a multiplicative inverse for any series $A(s) \in R[[s]]$ if and only if a_0 is a unit in R.

Multiplicative inverses

Let $A(s) = a_0 + a_1 s + a_2 s^2$ be a series over an integral domain, where a_0 is a unit. Find the coefficients necessary to construct $B(s)$, where $A(s)B(s) = 1$. We can construct them as follows:

- $a_0 b_0 = 1 \implies b_0 = 1/a_0$
- $a_0 b_1 + b_0 a_1 = 0 \implies b_1 = (b_0 a_1)/a_0$
- etc.

Ideals in $R[[s]]$

Let F be a field and let $F[[s]]$ be the set of formal power series in s over R. Then each ideal in $F[[s]]$ is generated by s^m for some m.
For each nonzero element $A(S)$ in $F[[s]]$, there exists m such that $a_m \neq 0$ and $a_n = 0$ for all $n < m$. Let M be the minimal such m over all the nonzero elements of $F[[s]]$. If we take an element $A(s)$ such that a_m is the first nonzero coefficient in the series, we can factor $A(s)$ into $s^M B(s)$, where $B(s)$ is series with nonzero constant term (an invertible series). Then $A(s)/B(s) = s^M$ and so s^M is in the ideal. By the minimality of M, s^M generates the ideal.

Returning to combinatorics

A *partition* of a number n is a set of positive integers that sum to n. We no longer have the restriction from 0 to 9 of the summands, and now we have that two sums that have the same summands listed in a different order are equivalent. The partitions of a few small numbers are as follows:

$$\begin{array}{ll} n=1 & 1 \\ n=2 & 2=1+1 \\ n=3 & 3=2+1=1+1+1 \\ n=4 & 4=3+1=2+2=2+1+1=1+1+1+1 \end{array}$$

Let p_n be the number of partitions of n. Then $p_0 = 1$, since the empty sum is a partition, and as we see from above $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, and $p_4 = 5$.

What is the generating function for p_n? Let $P_1(s)$ be the generating function for the number of partitions of n where each of the summands is 1. Since there is one such partition for each n, where $n \geq 0$,

$$P_1(s) = 1 + s + s^2 + \dots.$$

Multiplying each side of this equation by s yields

$$\begin{aligned} sP_1(s) &= s + s^2 + s^3 + \dots \\ &= P_1(s) - 1 \end{aligned}$$

which allows us to write

$$P_1(s) = \frac{1}{1-s}.$$

Let $P_2(s)$ be the generating function for the number of partitions of n where each of the summands is 2. For n even, there is one such partition of n. For n odd, there are no such partitions. Thus

$$P_2(s) = 1 + s^2 + s^4 + \dots.$$

Then we can multiply each side by s^2 to get

$$\begin{aligned} s^2 P_2(s) &= s^2 + s^4 + s^6 + \dots \\ &= P_2(s) - 1 \end{aligned}$$

and so

$$\begin{aligned} 1 &= (1 - s^2) P_2(s) \\ \frac{1}{1 - s^2} &= P_2(s). \end{aligned}$$

The partitions of n where the summands are all either 1 or 2 can be constructed as sums of partitions of k where the summands are all 1 and partitions of $n-k$ where the summands are all 2. Summing up over all possibilities for k, we find that the number of partitions of n with summands all equal to 1 or 2 is

$$\sum_{k=0}^{n} b_k c_{n-k},$$

where b_k is the number of partitions of k with summands all equal to 1 and $cn-k$ is the number of partitions of $n-k$ with summands all equal to 2. Therefore the generating function for the number of partitions of n where the summands are 1 or 2 is

$$P_1(s)P_2(s) = \frac{1}{(1-s)(1-s^2)}.$$

Repeating this process, we find that the generating function for the number of partitions of n where the summands are no greater than k is equal to

$$P_1(s)P_2(s)P_3(s)\dots P_k(s) = \frac{1}{(1-s)(1-s^2)(1-s^3)\dots(1-s^k)}.$$

The generating function for the number of partitions of n with no restrictions on n is thus

$$\begin{aligned} P(s) &= P_1(s)P_2(s)P_3(s)\dots \\ &= \frac{1}{(1-s)(1-s^2)(1-s^3)\dots}. \end{aligned}$$

Define the formal power series $Q(s)$ to be

$$Q(s) = (1-s)(1-s^2)\dots.$$

Let us consider how to find the coefficient q_k of $Q(s)$. Define the series

$$Q_k(s) = (1-s)(1-s^2)\dots(1-s^k)$$

and let q'_k be the coefficient of s^k in $Q_k(s)$. Notice that

$$\begin{aligned} Q_{k+1}(s) &= (1-s^{k+1})Q_k(s) \\ &= Q_k(s) - s^{k+1}Q_k(s) \end{aligned}$$

. Thus the coefficient of s^k in $Q_{k+1}(s)$ is also equal to q'_k. By repeating this process, we will find that the coefficient of s^k is unaffected by multiplications of $(1-s^{k+i})$ for $i > 0$. Hence $q_k = q'_k$.

We can find the coefficients up to q_k of $Q(s)$ by taking the coefficients from $Q_k(s)$. The first few terms of $Q(s)$ are as follows:

$$Q(s) = 1 - s - s^2 + s^5 + s^7 - s^12 - \dots. \tag{1}$$

An identity for $Q(s)$ from Euler:

$$Q(s) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(s^{\frac{3k^2-k}{2}} + s^{\frac{3k^2+k}{2}} \right).$$

We have that

$$P(s) = \frac{1}{Q(s)}, \qquad \text{which gives us that}$$
$$P(s)Q(s) = 1.$$

We already know that $p_0 q_0 = 1$. Then

$$\sum_{i=0}^{k} q_i p_{k-i} = 0$$

for all $k > 0$. This yields

$$0 = p_k + \sum_{i=1}^{k} q_i p_{k-i}$$
$$p_k = -\sum_{i=1}^{k} q_i p_{k-i}.$$

This yields the recurrence relation

$$p_n = p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + \dots,$$

which is a finite sum since $p_k = 0$ for all negative k.

Questions?