Partial, Total, and Lattice Orders in Group Theory

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Abstract

The algebraic structure of groups is familiar to anyone who has studied abstract algebra. Familiar to anyone who has studied any mathematics is the concept of ordering, for example the standard less than or equal to with the integers. We can combine these structures to create a new hybrid algebraic structure. In the following, we will define exactly how order structures and group structures interact, and then survey the results. We will examine how the order operation affects the elements of the group for various types of orders. We will generally explore partially ordered groups and their properties. We will also discuss lattice ordered groups in detail, exploring general properties such as the triangle inequality and absolute value in lattice ordered groups. Finally we will explore the theory of permutations, isomorphisms, and homomorphisms in lattice ordered groups.

1 Groups and Orders

Recall that a *relation* on a set X is a subset of $X \times X$. A relation $\preceq$ on X is a *partial order* if it satisfies the following:

1. For all $x \in X$, $x \preceq x$ (*reflexive*)
2. If $x \preceq y$ and $y \preceq x$, then $x = y$, for all $x, y \in X$ (*antisymmetric*)
3. If $x \preceq y$ and $y \preceq z$, then $x \preceq z$, for all $x, y, z \in X$ (*transitive*)

Furthermore, $\preceq$ is a *total order* when $x \preceq y$ and $y \preceq x$, then $x = y$, for all $x, y \in X$. This property is called *totality*, and if $\preceq$ has it we say $\preceq$ is *dichotomous*.

We call X together with a partial order $\preceq$ a *poset*. A poset is *directed upwards* if every pair of elements has an upper bound. If every pair of elements has a lower bound, then the poset is *directed downwards*. If a poset is both directed upwards and directed downwards, then we call the poset *directed*.

A poset X is *lattice* if every pair of elements in X has least upper bound and greatest lower bound under the partial order $\preceq$. In this case, we call $\preceq$ a *lattice-order*. These are the main order structures we will be discussing.

When we have a lattice, it should be noted that we can take advantage of the principle of duality. More formally, any statement that is true for lattices remains true when we replace $\preceq$ with $\succeq$ and $\vee$ and $\wedge$ are interchanged throughout the statement. This result will allow us to shorten some of our statements and proofs.

Recall further that a group is a a set G with a closed binary operation such that there is an identify element, every element has an inverse, and the operation is associative. We are now ready to combine the order structure with the group structure.

Definition 1.1. Let G be a group that is also a poset with partial order $\preceq$. Then G is *partially ordered group* if whenever $g \preceq h$ and $x, y \in G$, then $xgy \preceq xhy$. This property is called *translation-invariant*. We call G a *po-group*.

In a similar manner, a *directed* group is a po-group whose partial order is directed. A po-group whose partial order is a lattice is a *lattice-ordered group*, or $\mathcal{L}$-group. Additionally, if the order is total then G is a *totally ordered group*, or simply an *ordered group*.

Here we give a few examples to familiarize the concept of this structure.

Example 1.2. The additive groups of $\mathbb{Z}, \mathbb{R}$, and $\mathbb{Q}$ are all ordered groups under the usual ordering of less than or equal to.

Example 1.3. Let G be any group. Then G is *trivially ordered* if we define the order $\preceq$ by $g \preceq h$ if and only if $g = h$. With this order, then G is a partially ordered group.

Example 1.4. Every subgroup H of a partially ordered group G is a partially ordered group itself, where H inherits the partial order from G. The same is true for subgroups of ordered groups.

Note that a subgroup of a $\mathcal{L}$-group is not necessarily a $\mathcal{L}$-group. We will provide the necessary conditions soon.

Example 1.5. Let $\mathbf{V}$ be a vector space over the rationals, with basis $\{\mathbf{b}_i : i \in I\}$. Let $\mathbf{v}, \mathbf{w} \in \mathbf{V}$, with $\mathbf{v} = \sum_{i \in I} p_i \mathbf{b}_i$ and $\mathbf{w} = \sum_{i \in I} q_i \mathbf{b}_i$. Define $\mathbf{v} \preceq \mathbf{w}$ if and only if $q_i \leq r_i$ for all $i \in I$. Then $\mathbf{V}$ is a $\mathcal{L}$-group.

2 Po-groups

We now turn to some general properties of po-groups. First, a characterization of partial order-ability of a general group.

Proposition 2.1. *A group G can be partially ordered if and only if there is a subset P of G such that:*

1. $PP \subseteq P$

2. $P \cap P^{-1} = e$

3. *If* $p \in P$*, then* $gpg^{-1} \in P$ *for all* $g \in G$*.*

Proof. Let G be a po-group, and define $P = \{g \in G : e \preceq g\}$. Then if $g, h \in P$, $e \preceq g$. Then $eeh \preceq egh = gh$ since G is a po-group. Note that $e \preceq h = eeh$, so $gh \in P$ and $PP \subseteq P$. If $g \in P \cap P^{-1}$, then $e \preceq g$ and $e \preceq g^{-1}$, so $g = e$. Lastly, if $g \in G$ and $p \in P$, then $e = geg^{-1} \preceq gpg^{-1}$ since G is a po-group, so $gpg^{-1} \in P$.

Now let P be a subset, with these properties, of a group G. Define $g \preceq h$ if $hg^{-1} \in P$. Clearly $g \preceq g$ since $gg^{-1} = e \in P$. If $g \preceq h$ and $h \preceq k$ then $kg^{-1} = kh^{-1}hg^{-1} \in P$, so $g \preceq k$. If both $g \preceq h$ and $h \preceq g$, then hg^{-1} and $gh^{-1} \in P$, and both are in P^{-1}. Then $gh^{-1} = e$ by assumption on P, so $g = h$. Hence, $\preceq$ is a partial order.

If $x, y \in G$, and $g \preceq h$ for $g, h \in G$, then $hg^{-1} \in P$. Then $xhg^{-1}x^{-1} \in P$ by the properties of P, so $xhyy^{-1}g^{-1}x^{-1} \in P$. Therefore, $xgy \preceq xhy$, so G is a po-group under $\preceq$. □

The set $P = \{g \in G : e \preceq g\}$ is called the *positive cone* of G, and the elements of P are the *positive elements* of G. We call this set G^+. The set P^{-1} is called the *negative cone* of G. Positive cones determine everything about the order properties of a po-group.

There is a similar result on the total order-ability of groups.

Proposition 2.2. *A group* G *can be totally ordered if and only if there is a subset* P *such that:*

1. $PP \subseteq P$

2. $P \cap P^{-1} = e$

3. *If* $p \in P$*, then* $gpg^{-1} \in P$ *for all* $g \in G$*.*

4. $P \cup P^{-1} = G$

Proof. Let G be an ordered group. As before, let $P = \{g \in G : e \preceq g\}$. For all $g \in G$, either $e \preceq g$ or $g \preceq e$ since $\preceq$ is a total order. Then either $g \in P$ or $g \in P^{-1}$, so $G \subseteq P \cup P^{-1}$. Clearly $P \cup P^{-1} \subseteq G$, so by set equality $P \cup P^{-1} = G$.

Assume that P is a subset that has these properties. Now, as in proposition 2.1, defining $g \preceq h$ if $hg^{-1} \in P$ makes G a po-group. Since either $hg^{-1} \in P$ or $gh^{-1} \in P$ for all $g, h \in G$, either $g \preceq h$ or $h \preceq g$, and the ordering is total. Thus, G is an ordered group. □

There are further general results involving po-groups.

Proposition 2.3. *Let* G *be a po-group. Then* $g \preceq h$ *if and only if* $h^{-1} \preceq g^{-1}$

Proof. If $g \preceq h$, then $h^{-1}gg^{-1} \preceq h^{-1}hg^{-1}$, since G is a po-group. □

Proposition 2.4. *Let* G *be a po-group and* $g, h \in G$*.*

1. *If* $g \vee h$ *exists, then so does* $g^{-1} \wedge h^{-1}$*. Furthermore,* $g^{-1} \wedge h^{-1} = (g \vee h)^{-1}$

2. *If $g \wedge h$ exists, then so does $g^{-1} \vee h^{-1}$. Furthermore, $g^{-1} \vee h^{-1} = (g \wedge h)^{-1}$*

Proof. 1) Since $g \preceq g \vee h$, it follows that $(g \vee h)^{-1} \preceq g^{-1}$. Similarly, $(g \vee h)^{-1} \preceq h^{-1}$. If $f \preceq g^{-1}, h^{-1}$, then $g, h \preceq f^{-1}$. Then $g, h \preceq f^{-1}$, and so $g \vee h \preceq f^{-1}$. Therefore, $f \preceq (g \vee h)^{-1}$. Then by definition, $g^{-1} \wedge h^{-1} = (g \vee h)^{-1}$.

The proof of 2) is entirely similar, by the principle of duality. □

3 $\mathcal{L}$-groups

Now we take a closer look at $\mathcal{L}$-groups. We will develop a theory of the triangle inequality and absolute value along the way, as well as homomorphisms between $\mathcal{L}$-groups.

First, we note that every $\mathcal{L}$-group is directed. This follows the from the fact that every pair of elements in an $\mathcal{L}$-group has both a least upper bound and a greatest lower bound.

Before the following characterization, we need a definition.

Definition 3.1. Let G be a po-set. If every pair of elements have a least upper bound, then G is called an *upper semilattice*. If every pair of elements has greatest lower bound, then G is a lower semilattice. If G is both an upper semilattice and lower semilattice, then G is a lattice.

Proposition 3.2. *Let G be a po-group. Then the following are equivalent:*

1. *G is an $\mathcal{L}$-group.*
2. *G is directed and G^+ is a lattice.*
3. *For any $g \in G$, $g \vee e$ exists.*
4. *G is an upper semilattice.*

Proof. Clearly 1) implies 2) by our previous statement.

To show 2) implies 3), let $g^{-1} \preceq x$ and $e \preceq x$ for $g, x \in G$. Then since $e \preceq gx$, $gx \vee x$ exists. Then $g \vee e = (gx \vee x)x^{-1}$.

Now assume 3) and let $g, h \in G$. Then $g \vee e$ and $h \vee e$ exist. Let $x = gh^{-1} \vee e$. Then $xh = g \vee h$. Hence, G is an upper semilattice.

For 4) implies 1), let $g, h \in G$. Then let $x = g^{-1} \vee h^{-1}$, which exists by assumption. So, $x^{-1} = g \wedge h$. Hence, G is an $\mathcal{L}$-group. □

The following is an interesting characterization of the underlying lattice on an $\mathcal{L}$-group.

Theorem 3.3. *If G is an $\mathcal{L}$-group, then the lattice of G is distributive. In other words, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c, \in G$.*

Proof. Note that any lattice that satisfies the implication

$$\text{If } a \wedge b = a \wedge c \text{ and } a \vee b = a \vee c \text{ imply } b = c$$

is distributive. So under these assumptions, and using our previous results, $b = (a \wedge b)a^{-1}a(a \wedge b)^{-1}b = (a \wedge b)a^{-1}a(a^{-1} \vee b^{-1})b = (a \wedge b)a^{-1}(b \vee a) = (a \wedge c)a^{-1}(c \vee a) = (a \wedge c)a^{-1}a(a^{-1} \vee c^{-1})c = c$. □

4 Absolute Value in $\mathcal{L}$-groups

We turn now to absolute value and the triangle inequality in terms of $\mathcal{L}$-groups. Our goal is a characterization of abelian $\mathcal{L}$-groups.

Definition 4.1. For an $\mathcal{L}$-group G, and for $g \in G$:

1. The *positive part* of g, g_+, is $g \vee e$.
2. The *negative part* of g, g_-, is $g^{-1} \vee e$.
3. The *absolute value* of g, $|g|$, is $g_+ g_-$.

We summarize some general results.

Proposition 4.2. *Let G be an $\mathcal{L}$-group and let $g \in G$. Then:*

1. $g = g_+(g_-)^{-1}$
2. $g_+ \wedge g_- = e$

Proof. 1) $gg_- = g(g^-1 \vee e) = e \vee g = g_+$. So, $g = g_+(g_-)^{-1}$.
2) $g_+ \wedge g_- = gg_- \wedge g_- = (g \wedge e)g_- = (g \wedge e)(g^{-1} \vee e) = (g \wedge e)(g \wedge e)^{-1} = e$. $\square$

Proposition 4.3. *Let G be a $\mathcal{L}$-group and let $g, h \in G$. Then:*

1. $|g| = g_+ \vee g_- = g \vee g^{-1}$.
2. $|h| \preceq |g|$ *if and only if* $|g|^{-1} \preceq h \preceq |g|$.

Proof. 1) By proposition 4.2 $g_+ \wedge g_- = e$, so $|g| = g_+ g_- = g_+ \vee g_- = (g \vee e) \vee (g^{-1} \vee e) = g \vee g^{-1} \vee e = g \vee g^{-1}$, by the commutativity of the lattice on G.
2) If $|h| \preceq |g|$, then by the previous statement, $|h| = h \vee h^{-1}$. This implies that $h^{-1} \preceq |g|$, so $|g|^{-1} \preceq h$. Therefore, $|g|^{-1} \preceq h \preceq |g|$, as desired. $\square$

We are now in a position to state and prove the Triangle Inequality.

Theorem 4.4 (The Triangle Inequality)**.** *Let G be an $\mathcal{L}$-group. Then for all $g, h \in G$, $|gh| \preceq |g||h||g|$.*

Proof. Note that $gh \preceq |g||h| \preceq |g||h||g|$, and $|g|^{-1}|h|^{-1}|g|^{-1} \preceq gh$. Therefore, by proposition 4.3, $|gh| \preceq |g||h||g|$. $\square$

This is a fundamental result in $\mathcal{L}$-group theory. It may not seem as familiar or as practical as the usual Triangle Inequality in a traditional setting as there is a third term. However, with an extra condition on g and h, we can shorten it.

Proposition 4.5. *Let G be an $\mathcal{L}$-group, and let $g, h \in G$. If $gh = hg$, then $|gh| = |g||h|$.*

Proof. $|g||h| = (g \vee g^{-1})(h \vee h^{-1}) = g(h \vee h^{-1}) \vee g^{-1}(h \vee h^{-1}) = gh \vee gh^{-1} \vee g^{-1}h \vee g^{-1}h^{-1} = gh \vee gh^{-1} \vee hg^{-1} \vee h^{-1}g^{-1} = gh \vee h^{-1}g^{-1} = |gh|$. $\square$

Before we proceed, we need a lemma.

Lemma 4.6. *Let G be an $\mathcal{L}$-group, with $g \in G$. Then $|g^{-1}| = |g|$.*

Proof. $|g^{-1}| = (g^{-1} \vee e)(g \vee e) = g_- g_+ = g_+ g_- = |g|$. $\square$

With these tools, we can formulate a characterization of abelian $\mathcal{L}$-groups.

Theorem 4.7. *Let G be an $\mathcal{L}$-group. Then G is abelian if and only if for all pairs of elements $g, h \in G$, $|gh| \preceq |g||h|$.*

Proof. Note that if G is abelian, then by the the Triangle Inequality and proposition 4.5, we have already shown what we needed.

Suppose that for $g, h \in G$, $e \preceq g, h$. Then $|gh| = gh = |h^{-1}g^{-1}|$. Note that, by assumption and lemma 4.6, $|h^{-1}g^{-1}| \preceq |h^{-1}||g^{-1}| = |h||g| = hg$, implying that $gh \preceq hg$. Then, by symmetry, $hg \preceq gh$. Therefore, by the partial order on G, $gh = hg$, and thus the elements of G^+ commute.

Now for $x, y \in G$, $xy = x_+ x_-^{-1} y_+ y_-^{-1} = x_+ y_+ x_-^{-1} y_-^{-1} = y_+ x_+ y_-^{-1} x_-^{-1} = y_+ y_-^{-1} x_+ x_-^{-1} = yx$. Hence, G is abelian. $\square$

We have thus provided a sufficient characteristic of abelian $\mathcal{L}$-groups, using the triangle inequality. It is interesting to note that it was a property of the positive cone, G^+, of an $\mathcal{L}$-group that allowed us to discern this property.

5 Permutations, Homomorphisms, and Isomorphisms

In order talk about homomorphisms between $\mathcal{L}$-groups, we need to define functions between sets that preserve order. Let G and H be po-sets and $f : G \to H$ be a function. Then if whenever $g_1 \preceq g_2$ for $g_1, g_2 \in G$, then $f(g_1) \preceq f(g_2)$ in H, then f is *order preserving*. In this case f is called a *order homomorphism*, or a *ordermorphism*.

If G and H are lattices, a function $f : G \to H$ is a *lattice homomorphism* if for all $g_1, g_2 \in G$, $f(g_1 \vee g_2) = f(g_1) \vee f(g_2)$, and $f(g_1 \wedge g_2) = f(g_1) \wedge (g_2)$. If f is additionally bijective, then f is a *lattice isomorphism*. It is true that if f is a lattice homomorphism, then it is also an ordermorphism. Furthermore, if f is a lattice isomorphism, then, naturally, f^{-1} is a also a lattice isomorphism.

As is the case with group automorphisms, the set of all lattice automorphisms of a lattice G forms a group. In this case, the operation is composition of functions.

Now we return to $\mathcal{L}$-groups. A lattice automorphism, σ, of an $\mathcal{L}$-group is called an *$\mathcal{L}$-permutation* if it maps the identity to the identity (i.e. $\sigma(e) = e$). Finally, if G and H are $\mathcal{L}$-groups and σ is a function given by $\sigma : G \to H$ where σ is both a lattice homomorphism and a group homomorphism, then σ is an *$\mathcal{L}$-homomorphism*. Note that many theorems on group homomorphisms translate nicely into theorems on $\mathcal{L}$-homomorphisms, because of the way $\mathcal{L}$-homomorphisms are constructed. Notable are the three Isomorphism Theorems.

For now we provide a useful description of $\mathcal{L}$-homomorphisms. But first, a lemma.

Lemma 5.1. *Let G be an $\mathcal{L}$-group. Then for any elements g and $h \in G$, $g \wedge h = e$ if and only if $gh = |gh^{-1}|$.*

Proof. Assume that $g \wedge h = e$. It follows that $gh = |gh^{-1}|$. Suppose on the other hand that, $gh = |gh^{-1}|$. Note that $|gh^{-1}| = (hg^{-1} \vee e)(gh^{-1} \vee e) = (g \vee h)h^{-1} \vee (g \vee h)g^{-1} = (g \vee h)(g^{-1} \vee h^{-1}) = (g \vee h)(g \wedge h)^{-1}$. So, since $g \vee h \preceq (g \vee h)(g \wedge h)^{-1}$, $g \wedge h = e$. □

Theorem 5.2. *Let G and H be $\mathcal{L}$-groups, and $\sigma : G \to H$ a group homomorphism. Then the following are equivalent:*

1. *σ is a $\mathcal{L}$-homomorphism.*

2. *$\sigma(|g|) = |\sigma(g)|$, for all $g \in G$.*

3. *If $g \wedge H = e$ for $g, h \in G$, then $\sigma(g) \wedge \sigma(h) = e$*

4. *$(\sigma(g))_+ = \sigma(g_+)$ for all $g \in G$.*

Proof. Assume that σ is an $\mathcal{L}$-homomorphism. Then $\sigma(|g|) = \sigma(g_+g_-) = \sigma(g_-)\sigma(g_+) = [\sigma(g \vee e)][\sigma(g^{-1} \vee e)] = (\sigma(g) \vee e)(\sigma(g^{-1}) \vee e) = \sigma(g)_+\sigma(g^{-1})_- = |\sigma(g)|$.

Next assume 2), and let $g \vee h = e$. Then $|\sigma(g)\sigma(h)^{-1}| = |\sigma(g)\sigma(h^{-1})| = |\sigma(gh^{-1})|= \sigma(|gh^{-1}| = \sigma(gh) = \sigma(g)\sigma(h)$, so by lemma 5.1, $\sigma(g) \wedge \sigma(h) = e$.

Now assume that 3) holds. If $g_+ \wedge g_- = e$, then $\sigma(g_+) \wedge \sigma(g_-) = e$. Therefore, $(\sigma(g))_+ = [(\sigma(g_+))(\sigma(g_-))^{-1}]_+ = \sigma(g_+)$.

Finally, assume 4). Then $\sigma(g \vee h) = \sigma(h(gh^{-1} \vee e)) = \sigma(h)\sigma(gh^{-1} \vee e) = \sigma(h)(\sigma(gh^{-1} \vee e) = \sigma(h)(\sigma(g)\sigma(h^{-1}) \vee e) = \sigma(g) \vee \sigma(h)$. A similar argument shows that $\sigma(g \wedge h) = \sigma(g) \wedge \sigma(h)$. Therefore, σ is a lattice homomorphism, and is thus an $\mathcal{L}$-homomorphism. □

In applications, part 4 of the previous theorem is the most useful to verify that a group homomorphism is indeed a $\mathcal{L}$-homomorphism. At this point we note that the induced homomorphism theory carries over nicely to $\mathcal{L}$-homomorphisms.

6 Subgroups

We close with a look at subgroups of $\mathcal{L}$-groups. A *sublattice* of a lattice L is a subset S such that S is also a lattice with the ordering inherited from L. A subgroup of S of an $\mathcal{L}$-group G is an $\mathcal{L}$-*subgroup* if S is also a sublattice of G. Note this slight difference as opposed to subgroups of po-groups earlier.

Furthermore, $\mathcal{L}$-subgroups of $\mathcal{L}$-subgroups are themselves $\mathcal{L}$-subgroups. For example, if A is an $\mathcal{L}$-subgroup of B which is an $\mathcal{L}$-subgroup of G, then A is an $\mathcal{L}$-subgroup of G. Additionally, the intersection of $\mathcal{L}$-subgroups is again an $\mathcal{L}$-subgroup.

A fact that carries over from group theory is that the kernel of an $\mathcal{L}$-homomorphism is an $\mathcal{L}$-subgroup.

With an ordering on a group, we can describe new kinds of subgroups.

Definition 6.1. A subset S of a po-group G is *convex* if whenever $s, t \in S$ and $s \preceq g \preceq t$ in T, then $g \in S$.

This definition gives rise to *convex subgroups* of a po-groups and *convex* $\mathcal{L}$-subgroups of $\mathcal{L}$-groups.

With convex subgroups in hand, we can define coset orderings.

Definition 6.2. Let G be a po-group with partial order $\leq$ and S a convex subgroup of G. Let $\mathcal{R}(S)$ be the set of right cosets of S in G. On $\mathcal{R}(S)$, define $Sx \preceq Sy$ if there exists an $s \in S$ such that $sy \leq x$, for $x, y \in G$. Then $\preceq$ is partial ordering of $\mathcal{R}(S)$, and it is called the *coset ordering* of $\mathcal{R}(S)$.

This is technically the *right* coset ordering. An entirely similar definition may be made for left cosets.

We end with a particularly pleasant theorem about convex $\mathcal{L}$-subgroups and the coset ordering.

Theorem 6.3. *Let G be a $\mathcal{L}$-group. Then a subgroup S of G is a convex $\mathcal{L}$-subgroup if and only if $\mathcal{R}(S)$ is a distributive lattice under the coset ordering.*

Proof. Let S be a convex $\mathcal{L}$-subgroup. Then $Sx \preceq S(x \vee y)$ and $Sy \preceq (x \vee y)$. Suppose that $Sx \preceq Sd$ and $Sy \preceq Sd$. There exists $s_1, s_2 \in S$. such that $s_1 x \preceq d$ and $s_2 y \preceq d$, by the definition of the coset ordering. Then $(s_1 \wedge s_2)x \preceq d$ and $(s_1 \wedge s_2)y \preceq d$. This implies that $(s_1 \wedge s_2)(x \vee y) \preceq d$. Thus, $S(x \vee y) \preceq Sd$, so $S(x \vee y) = Sx \vee Sy$. An entirely similar argument show that $S(x \wedge y) = Sx \wedge S_y$. Thus, $\mathcal{R}(S)$ is a lattice under $\preceq$.

Note that $Sx \wedge (Sy \vee Sz) = Sx \wedge S(y \vee z) = S[x \wedge (y \vee z)] = S[(x \wedge y) \vee (x \wedge z)] = S(x \wedge y) \vee S(x \wedge z) = (Sx \wedge Sy) \vee (Sx \wedge Sz)$. The principle of duality guarantees we can flip $\wedge$ and $\vee$. Hence, $\mathcal{R}(S)$ is distributive.

On the other hand, suppose that $|g| \preceq |s|$ where $s \in S$. Then $s \wedge s^{-1} \preceq |s|^{-1} \preceq g \preceq |s| = s \vee s^{-1}$. Since $S = Ss \wedge Ss^{-1} \preceq Sg \preceq Ss \vee Ss^{-1} = S$, so $g \in S$. Therefore, S is a convex $\mathcal{L}$-subgroup of G. $\square$

7 Conclusion

Thus concludes our survey of ordered structures in group theory. We have seen how natural it is for an order relation to interact with the elements of a group. We have provided useful characterizations of po-groups and $\mathcal{L}$-groups in order to grasp their underlying structure. Finally, we have seen how the familiar concepts of homomorphisms, subgroups, and cosets from group theory can be applied to ordered groups.

References

[1] Darnel, Michael R. *Theory of Lattice-ordered Groups*. New York: Marcel Dekker Inc.,1995.

[2] Glass, A. M. W. *Partially Ordered Groups*. River Edge: World Scientific, 1999.

[3] Holland, W. Charles. *Ordered Groups and Infinite Permutation Groups*. Dordrecht: Kluwer Academic Publishers, 1996.

[4] Schwartz, Niels, and Madden, James J. *Semi-algebraic Function Rings and Reflectors of Partially Ordered Rings*. New York: Springer, 1999.

[5] Steinberg, Stuart A. *Lattice-ordered Rings and Modules*. New York: Springer, 2010.