Polynomial Resultants

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The Resultant

Definition:

For $f(x) = a_n x^n + ... + a_1 x + a_0$,

$g(x) = b_m x^m + ... + b_1 x + b_0 \in F[x]$,

$$\text{Res}(f, g, x) = a_n^m b_m^n \prod_{i,j} (\alpha_i - \beta_j),$$

where $f(\alpha_i) = 0$ for $1 \leq i \leq n$, and $g(\beta_j) = 0$ for $1 \leq j \leq m$.

Common Factor Lemma

Let $f(x), g(x) \in F[x]$ have degrees n and m, both greater than zero, respectively. Then $f(x)$ and $g(x)$ have a non-constant common factor if and only if there exist nonzero polynomials $A(x), B(x) \in F[x]$ such that $\deg(A(x)) \leq m - 1$, $\deg(B(x)) \leq n - 1$ and $A(x)f(x) + B(x)g(x) = 0$.

Proof

($\Longrightarrow$)
Assume $h(x) \in F[x]$ is a common factor of $f(x)$ and $g(x)$, then $f(x) = h(x)f_1(x)$ and $g(x) = h(x)g_1(x)$.
Consider,

$$\begin{aligned} &g_1(x)f(x) + (-f_1(x))g(x) \\ &= g_1(x)(h(x)f_1(x)) - f_1(x)(h(x)g_1(x)) \\ &= 0 \end{aligned}$$

Proof

$(\Longleftarrow)$
Assume $A(x)$ and $B(x)$ exist.
Assume, contrary to the lemma, that $f(x)$ and $g(x)$ share no non-constant factors. Then
$\gcd(f(x), g(x)) = r(x)f(x) + s(x)g(x) = 1$

Let

$$f(x) = a_n x^n + ... + a_1 x + a_0,\ a_n \neq 0$$

$$g(x) = b_m x^m + ... + b_1 x + b_0,\ b_m \neq 0$$

$$A(x) = c_{m-1} x^{m-1} + ... + c_1 x + c_0,$$

$$B(x) = d_{n-1} x^{n-1} + ... + d_1 x + d_0.$$

And $A(x)f(x) + B(x)g(x) = 0$

The Sylvester Matrix

$$\mathrm{Syl}(f, g, x)$$

$$\begin{bmatrix}
a_n & & & & b_m & & & \\
a_{n-1} & a_n & & & b_{m-1} & b_m & & \\
a_{n-2} & a_{n-1} & \ddots & & b_{m-2} & b_{m-1} & \ddots & \\
\vdots & \vdots & \ddots & a_n & \vdots & \vdots & \ddots & b_m \\
\vdots & \vdots & & a_{n-1} & \vdots & \vdots & & b_{m-1} \\
a_0 & a_1 & & & b_0 & b_1 & & \\
 & a_0 & \ddots & \vdots & & b_0 & \ddots & \vdots \\
 & & \ddots & a_1 & & & \ddots & b_1 \\
 & & & a_0 & & & & b_0
\end{bmatrix}$$

Properties of the Sylvester Matrix

- The determinant of the Sylvester matrix $\mathrm{Syl}(f, g, x)$ is a polynomial in the coefficients a_i,b_j of the polynomials $f(x)$ and $g(x)$. Further,

$$\det(\mathrm{Syl}(f, g, x)) = \mathrm{Res}(f, g, x).$$

- For $f(x), g(x) \in F[x]$, there exist polynomials $A(x), B(x) \in F[x]$ so that $A(x)f(x) + B(x)g(x) = \mathrm{Res}(f, g, x)$.

Applications: The Discriminant

For a polynomial $f(x) \in F[x]$, where
$f(x) = a_n x^n + \dots + a_1 x + a_0$,
the discriminant is given by

$$D = \frac{(-1)^{n(n-1)/2}}{a_n} \text{Res}(f, f', x),$$

where $f'(x)$ is the derivative of $f(x)$.

Discriminant Example

Let $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

$$D = \frac{(-1)^{2(2-1)/2}}{a} \begin{vmatrix} a & 2a & 0 \\ b & b & 2a \\ c & 0 & b \end{vmatrix} = \frac{-1}{a}(a(b^2) - b(2ab) + c(4a^2))$$

$$= \frac{-1}{a}(ab^2 - 2ab^2 + 4a^2c)$$

$$= \frac{-1}{a}(-ab^2 + 4a^2c)$$

$$= b^2 - 4ac,$$

Applications: Elimination

$\text{Res}_i : F[x_1, ..., x_n] \times F[x_1, ..., x_n] \to F[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$, where Res_i is the resultant relative to the variable x_i.

Elimination Example

Let $f(x, y) = x^2y^2 - 25x^2 + 9$ and $g(x, y) = 4x + y$ be two polynomials in $F[x, y]$.

Partial Solutions

Theorem:
If $(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n)$ is a solution to a homogeneous system of polynomials in $F[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$ obtained by taking resultants of polynomials in $F[x_1, ..., x_n]$ with respect to x_i, then there exists $\alpha_i \in E$, where E is the field in which all polynomials in the system split, such that $(\alpha_1, ..., \alpha_i, ..., \alpha_n)$ is a solution to the system in $F[x_1, ..., x_n]$.

The End

Thank You.
This is the end.

Questions?
Comments?