Math 290
Exam 1
Chapter SLE

Name: Key

Dr. Beezer
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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= -2 \\ x_2 - 2x_3 + x_4 &= 1 \\ 2x_1 + 3x_2 - 4x_3 + 7x_4 &= 5 \end{aligned}$$

Form the augmented matrix and row-reduce (w/ Sage)

$$\left[\begin{array}{cccc|c} 1 & -2 & 5 & 0 & -2 \\ 0 & 1 & -2 & 1 & 1 \\ 2 & 3 & -4 & 7 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since column 5 is a pivot column, by Theorem RCLS the system is inconsistent.

Answer:

$\emptyset = \{\ \}$

2. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} x_1 - 2x_2 - 4x_3 + 3x_4 + 4x_5 &= 1 \\ x_2 + 2x_3 - 2x_4 - x_5 &= 1 \\ -x_1 + 3x_2 + 6x_3 - 5x_4 - 5x_5 &= 0 \\ 2x_1 - 3x_2 - 6x_3 + 4x_4 + 7x_5 &= 3 \end{aligned}$$

Augmented matrix, row-reduce w/ Sage

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 2 & 3 \\ 0 & 1 & 2 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Consistent by Theorem RCLS.
$r=2$ pivot columns.
x_3, x_4, x_5 free

$x_1 = 3 + x_4 - 2x_5$
$x_2 = 1 - 2x_3 + 2x_4 + x_5$

Answer:

$$S = \left\{ \begin{bmatrix} 3 + x_4 - 2x_5 \\ 1 - 2x_3 + 2x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle|\, x_3, x_4, x_5 \in \mathbb{C} \right\}$$

3. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (20 points)

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 7 & 1 \\ 3 & 2 & 12 & 1 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 6 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Determine if the matrix below is nonsingular or singular. Explain your reasoning carefully and thoroughly. (15 points)

$$\begin{bmatrix} 3 & 4 & -1 & 6 & 8 \\ 0 & 1 & -1 & -1 & -5 \\ 2 & 5 & -2 & 4 & -1 \\ 4 & 4 & -1 & 6 & 7 \\ 2 & 3 & -1 & 4 & 5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5$$

By Theorem NMRRI the matrix is nonsingular.

5. Say as much as possible about the solution set of each system, along with justifications for your answers. (15 points)

   (a) Homogeneous, 5 variables and 8 equations.

   $\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ <u>is</u> a solution. We can't say more.

   (b) Coefficient matrix is nonsingular.

   By Theorem NMUS, the solution is unique.

   (c) 8 variables, 5 equations.

   inconsistent

   OR

   consistent, and then Theorem CMVEI there would be infinitely many solutions.

6. Suppose that the coefficient matrix of a homogeneous system has two columns that are identical. Prove that the system has infinitely many solutions. (15 points)

Suppose columns k & l are identical.

1) Homogeneous ⇒ one solution, at least.

Show that $x_1 = x_2 = \cdots = x_{k-1} = 0$, $x_k = 1$, $x_{k+1} = \cdots = x_{l-1} = 0$, $x_l = -1$,

$x_{l+1} = \cdots = x_n = 0$

is a second solution, hence infinitely many.

<u>OR</u>

2) Row-reduce the matrix & columns k & l will remain equal. They cannot <u>both</u> be pivot columns, so there is at least one non-pivot column. Hence a free variable, hence infinitely many solutions (since consistent).