Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. For the matrix A below, compute the inverse, or explain how you know A does not have an inverse. (15 points)

$$A = \begin{bmatrix} -1 & 2 & 0 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & -1 & 3 & -4 \\ -1 & 1 & 2 & -5 \end{bmatrix}$$

$$[A \mid I_4] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 4 & 4 & -2 \\ 0 & 1 & 0 & 0 & 2 & 1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}\right]$$

A nonsingular (NMRRI)

CINM + OSIS $\Rightarrow A^{-1}$

Answer:

$$\begin{bmatrix} 1 & 4 & 4 & -2 \\ 2 & 1 & 2 & -2 \\ 2 & -1 & 1 & -2 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

2. A matrix A has the extended echelon form given below. A vector of constants $\mathbf{b}$ is given. Decide if the linear system $\mathcal{LS}(A, \mathbf{b})$ is consistent or inconsistent, with a careful explanation. Most of the points for this problem will come from the quality of the explanation. (20 points)

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & -3 & 0 & -1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 1 & 2 & -2 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right] \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

A has 4 rows, hence augmented with I_4

then 2 zero rows so $L = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$LS(A, \underset{\sim}{b})$ consistent $\iff \underset{\sim}{b} \in C(A) \iff \underset{\sim}{b} \in N(L)$

Theorem CSCS

Theorem FS; $C(A) = N(L)$

Lets check

$$L\underset{\sim}{b} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq \underset{\sim}{0}, \text{ so } \underset{\sim}{b} \notin N(L)$$

and the system is inconsistent

3. Consider the matrix B. In each part, find a set of vectors whose span is the column space of A, $\mathcal{C}(A)$, and meets the additional requirements, and restrictions on techniques used. Be certain to explain the theorems and definitions employed. (35 points)

$$B = \begin{bmatrix} -7 & 28 & -11 & 19 & 2 & 12 & -8 \\ 11 & -44 & 29 & -65 & -7 & -50 & 32 \\ 4 & -16 & 15 & -37 & -4 & -30 & 19 \\ 1 & -4 & 16 & -46 & -5 & -40 & 25 \\ 0 & 0 & -3 & 9 & 1 & 8 & -5 \end{bmatrix}$$

$$[B \mid I_5] \xrightarrow{RREF} \left[\begin{array}{ccccccc|ccccc} 1 & -4 & 0 & 2 & 0 & 2 & -1 & 0 & 0 & 1 & -3 & -11 \\ 0 & 0 & 1 & -3 & 0 & -2 & 1 & 0 & 0 & -1 & 4 & 16 \\ 0 & 0 & 0 & 0 & 1 & 2 & -2 & 0 & 0 & -3 & 12 & 49 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \end{array}\right]$$

(a) Use only the definition of a column space.

The column space is the span of the columns so just list them all, Definition CSM

Answer:

$$\left\{ \begin{bmatrix} -7 \\ 11 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 28 \\ -44 \\ -16 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 29 \\ 15 \\ 16 \\ -3 \end{bmatrix}, \begin{bmatrix} 19 \\ -65 \\ -37 \\ -46 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ -4 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 12 \\ -50 \\ -30 \\ -40 \\ 8 \end{bmatrix}, \begin{bmatrix} -8 \\ 39 \\ 19 \\ 25 \\ -5 \end{bmatrix} \right\}$$

(b) The set is linearly independent and each vector in the set is a column of B.

Pivot columns $= 1, 3, 5$

Theorem BCS says we can use the original columns w/ same indices

Answer:

$$\left\{ \begin{bmatrix} -7 \\ 11 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 29 \\ 15 \\ 16 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ -4 \\ -5 \\ 1 \end{bmatrix} \right\}$$

(c) The set is linearly independent and is constructed using theorems about row spaces in non-trivial ways.

Defn RSM $\quad \mathcal{C}(B) = \mathcal{R}(B^t) = \mathcal{R}(\text{a matrix row-equivalent to } B^t)$

$$B^t \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For linear independence do not include zero rows, Theorem BRS

Answer:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(d) The set is linearly independent and is constructed using Theorem FS ("Four Subspaces") in a non-trivial way.

$\mathcal{C}(B) \underset{\uparrow \text{ Theorem FS}}{=} \mathcal{N}(L) \qquad L = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & -3 & 1 & 0 \end{bmatrix}$

Theorem BNS, "x_3, x_4, x_5 free"

$$\begin{bmatrix} \\ \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Answer:

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Suppose that A is an $m \times n$ matrix. Prove that $(\alpha A)^t = \alpha A^t$. (15 points)

Prove two matrices are equal with Defn ME. Notice that these are $\underline{n \times m}$ matrices. So for $1 \leq i \leq n$ & $1 \leq j \leq m$,

$$
\begin{aligned}
[(\alpha A)^t]_{ij} &= [\alpha A]_{ji} && \text{Defn TM} \\
&= \alpha [A]_{ji} && \text{Defn MSM} \\
&= \alpha [A^t]_{ij} && \text{Defn TM} \\
&= [\alpha A^t]_{ij} && \text{Defn MSM}
\end{aligned}
$$

Thus, the matrices $(\alpha A)^t$ & αA^t are equal.

5. Suppose that Q is an $n \times n$ unitary matrix and $\mathbf{x} \in \mathbb{C}^n$ is any vector. Prove that $\|Q\mathbf{x}\| = \|\mathbf{x}\|$. (This is the statement of a theorem in the book, so do not quote any part of that theorem as part of your answer.) (15 points)

$$
\begin{aligned}
\|Q\mathbf{x}\| &= \sqrt{\langle Q\mathbf{x}, Q\mathbf{x}\rangle} && \text{Theorem IPN} \\
&= \sqrt{(Q\mathbf{x})^* Q\mathbf{x}} && \text{Theorem MMIP} \\
&= \sqrt{(\mathbf{x}^* Q^*)(Q\mathbf{x})} && \text{Theorem MMT, MMCC} \\
&= \sqrt{\mathbf{x}^*(Q^*Q)\mathbf{x}} && \text{Theorem MMA} \\
&= \sqrt{\mathbf{x}^* I \mathbf{x}} && \text{Defn UM} \\
&= \sqrt{\mathbf{x}^* \mathbf{x}} && \text{Theorem MMIM} \\
&= \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle} && \text{Theorem MMIP} \\
&= \|\mathbf{x}\| && \text{Theorem IPN}
\end{aligned}
$$