Universal Algebra

and an

Introduction to Model Theory

Jesse Jenks

April 2019

Abstract Algebra
Math 434
University of Puget Sound

Abstract

While taking an undergraduate abstract algebra course, students are exposed to a wide range of algebraic structures. After seeing several formalization of these structures, it is natural to wonder if there is a more general theory relating these structures. This paper discusses the basics of Universal Algebra from the perspective of generalizing the definitions and results seen in an abstract algebra course.

Table of Contents

1 Introduction and Motivation

After taking several upper level mathematics courses, it can begin to feel like there is a pattern in the way topics are formalized. For example, as Enderton says in [3],

> Many a mathematics class, on its first day, begins with the instructor saying something like one of the following: [...]
>
> - "A group is defined to consist of a nonempty set G together with a binary operation $\circ$ satisfying the axioms. ..."
> - "A [...] field is defined to consist of a nonempty set F together with two binary operations $+$ and $\cdot$ [...] satisfying the axioms. ..."
> - "A vector space is defined to consist of a nonempty set V together with a binary operation $+$ and, for each [complex] number r, an operation called scalar multiplication such that. ..."

This paper serves as an introduction to Universal Algebra for students who are curious about this a more general theory of algebra and algebraic structures.

2 Background

The kernel of a function is typically defined as the inverse image of $\mathbf{0}$, whatever $\mathbf{0}$ may refer to in that context. For example in linear algebra, the kernel of a matrix M is the set of vectors $\mathbf{v}$ such that $M\mathbf{v} = \mathbf{0}$, the zero vector. In group and ring theory, the kernel of a homomorphism ϕ is the set of elements a such that $\phi(a) = 0$, the additive identity. However in order to be as general as possible, we will expand our definition of a kernel so as not assume the existence of an identity element.

Definition 1. (Kernel) The *kernel* of a function $\varphi : A \to B$ is the set

$$\{\langle a, b\rangle \in A^2 \mid \varphi(a) = \varphi(b)\}.$$

We denote the kernel of φ by $\ker(\varphi)$.

Note. The reader is assumed to be familiar with equivalence relations and equivalence classes.

Definition 2. (Equivalence Class) Let θ be an equivalence relation on A. Then for a given $a \in A$, we write a/θ to denote the set

$$\{b \in A \mid \langle a, b\rangle \in \theta\}.$$

This set is is called the *equivalence class of a by θ*.

Definition 3. (Quotient set) Let θ be an equivalence relation on A. The set of all equivalence classes is

$$A/\theta = \{a/\theta \mid a \in A\}.$$

This set is called the *quotient set of A by θ*.

Theorem 1. *The kernel of a function is an equivalence relation.*

Proof. Let $\varphi : A \to B$ be a function. Then for all $a, b, c \in A$,

- $\varphi(a) = \varphi(a)$ (reflexivity).
- $\varphi(a) = \varphi(b) \Rightarrow \varphi(b) = \varphi(a)$ (symmetry).
- If $\varphi(a) = \varphi(b)$ and $\varphi(b) = \varphi(c)$, then $\varphi(a) = \varphi(c)$ (transitivity).

This shows that $\ker(\varphi)$ is an equivalence relation. $\square$

Definition 4. (Canonical Map) The canonical map for an equivalence relation θ is the map from A to A/θ given by

$$a \mapsto a/\theta.$$

We are now ready to prove an important theorem.

Theorem 2. *If $\psi : A \to B$ is a function with $K = \ker(\psi)$, then K is an equivalence relation on A. Let $\phi : A \to A/K$ be the canonical map. Then there exists a unique bijection $\eta : A/K \to \psi(A)$ such that $\psi = \eta \circ \phi$.*

Proof. (following [6] Theorem 11.10) We know that $\ker(\psi)$ is an equivalence relation on A. Define η by $\eta(a/K) = \psi(a)$. We first show that η is a well-defined map. If $a_1/K = a_2/K$, then $\psi(a_1) = \psi(a_2)$; consequently,

$$\eta(a_1/K) = \psi(a_1) = \psi(a_2) = \eta(a_2/K).$$

Thus, η does not depend on the choice of equivalence class representatives and the map $\eta : A/K \to \psi(A)$ is uniquely defined since $\psi = \eta \circ \phi$. Clearly, η is onto $\psi(A)$. To show that η is one-to-one, suppose that $\eta(a_1/K) = \eta(a_2/K)$. Then $\psi(a_1) = \psi(a_2)$. This implies that $a_1/K = a_2/K$, so η is one-to-one. $\square$

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & B \\ & A/\ker(\psi) \arrow[ur, "\eta"'] & \end{tikzcd}$$

This the the most general form of the First Isomorphism Theorem. When the isomorphism theorems are first introduced in the context of groups and rings, it can feel like it "just so happens" that there are similar isomorphism theorems in both theories, perhaps due to the fact that these structures share a group operation. However, this theorem shows that the connection runs much deeper, and is more general.

3 Universal Algebra

In a course on abstract algebra, we often study what are sometimes called *models of theories*. For example, we might study the *Theory of Groups* and several models of this theory, like $\mathbb{Z}_6$ or D_4. Eventually we encounter an even wider range of algebraic theories, like Ring theory and Field theory. In Universal Algebra, we take a further step back, and study all of these algebraic structures.

3.1 Algebras

Definition 5. (Operations) An n-ary operation, or function on a nonempty set A is any function mapping A^n to A, where n is a non-negative integer. We define A^0 to be the set containing the empty set $\{\emptyset\}$. In this special case, the domain of a *nullary* function contains only a single element, and thus we can identify this function with its image in A. For this reason, we can call this kind of function a *constant* function.

Definition 6. (Signature) A *signature* for an algebra is a set $\mathscr{F}$ of function symbols. Each symbol f is assigned a positive integer n called the *arity* of f. In the context of universal algebra, signatures are often called *types*.

Definition 7. (Algebra) An *Algebra* $\mathbf{A}$ with signature $\mathscr{F}$ and *universe* A is a pair $\langle A, F\rangle$, where F is a set of operations on A with corresponding symbols in $\mathscr{F}$. Usually F is a finite set $\{f_1^{\mathbf{A}}, \dots, f_k^{\mathbf{A}}\}$, so we write $\langle A, f_1^{\mathbf{A}}, \dots, f_k^{\mathbf{A}}\rangle$.

Note. (On Notation) For an algebra $\mathbf{A} = \langle A, F\rangle$ with signature $\mathscr{F}$, we distinguish the function *symbol* $f \in \mathscr{F}$ with the "actual" function $f^{\mathbf{A}} \in F$ with a superscript. This distinction will become important later when discussing multiple algebraic structures with the same signature.

Example 1. In group theory, we might study the additive group $\mathbb{Z}_3$. In this case, our universe G is $\{0, 1, 2\}$, and has a signature composed of three operation. These operations are $\cdot$ (the group operation), $^{-1}$ (inverse operation), and $\mathbf{1}$ (identity element). These operations have arity 2, 1, and 0 respectively. Note that $\mathbf{1}$ is *the* function $\emptyset \mapsto 0$.

Definition 8. (Congruence) Let $\mathbf{A}$ be an algebra and $\sim$ an equivalence relation on the universe of $\mathbf{A}$. The relation $\sim$ is a *congruence on* $\mathbf{A}$ if for every n-ary operation $f^{\mathbf{A}}$, and elements $a_i, b_i \in A$ such that $a_i \sim b_i$ for $1 \le i \le n$,

$$f^{\mathbf{A}}(a_1, \dots, a_n) \sim f^{\mathbf{A}}(b_1, \dots, b_n).$$

In other words, the equivalence relation "respects" the operations on the algebra.

Definition 9. Let $\mathbf{A}$ be an algebra, and $\sim$ a congruence on $\mathbf{A}$. The *quotient algebra of* $\mathbf{A}$ *by* $\sim$, written $\mathbf{A}/{\sim}$ is an algebra with the same signature as $\mathbf{A}$ whose universe is the quotient set $A/{\sim}$ and whose operations satisfy

$$f^{\mathbf{A}/\sim}(a_1/{\sim}, \dots, a_n/{\sim}) = f^{\mathbf{A}}(a_1, \dots, a_n)/{\sim}.$$

Example 2. One of the first examples of congruence classes are cosets of a subgroup. For example take the following two cosets of the subgroup $H = \{0, 4, 8\}$ of $\mathbb{Z}_{12}$ under addition, $\{0, 4, 8\}$ and $\{3, 7, 11\}$. Then the relation $a \sim b$ if a and b are in the same coset is a congruence on $\mathbb{Z}_{12}$ Grab 0, 8, 3, and 7. Then $0 \sim 8$, $7 \sim 3$ and $0 + 7 = 7$, $8 + 3 = 11$, and $7 \sim 11$. Thus we have

$$7 = (0 + 7) \sim (8 + 3) = 11.$$

Now if we consider the quotient group $\mathbb{Z}_{12}/{\sim}$,

$$(0/{\sim}) + (3/{\sim}) = (0 + 3)/{\sim}.$$

In the normal course of studying group theory however, this is usually written

$$(0 + H) + (3 + H) = (0 + 3) + H.$$

With groups and rings it is easier to see how our more general definition of an algebra corresponds to the usual definition. But what about modules or vector spaces?

Example 3. (The signature for a Module over a ring R) A Module with universe M, has the same operations as a group: a binary operation $+$, a unary operation $^{-1}$, and a nullary operation $\mathbf{0}$. But modules also have a special operation called scalar multiplication. We usually denote scalar multiplication as a function like $R \times M \to M$, but our definition of an algebra only allows functions like $M^n \to M$. So should we change our definition? Should we not consider a module an algebra? Neither of these solutions is particularly appealing.

Recall that in the definition of a signature, Definition 6, it was noted that there are usually only finitely many operations. But this is not necessary. In fact, there is a clever way we can use this to our advantage. The idea is to add a function for each scalar. For each element $r \in R$, add to our signature a function $f_r : M \to M$, which defines scalar multiplication of any element in M by r. So a module over a ring R is an algebra with the signature

$$\{+, ^{-1}, \mathbf{0}, \{f_r\}_{r \in R}\}.$$

3.2 Homomorphisms

Definition 10. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras with the same signature $\mathscr{F}$. If there is a function $\varphi : A \to B$ such that for every function symbol f of arity n in $\mathscr{F}$ and every $a_1, \ldots, a_n \in A$,

$$\varphi\left(f^{\mathbf{A}}(a_1, \ldots, a_n)\right) = f^{\mathbf{B}}\left(\varphi(a_1), \ldots, \varphi(a_n)\right),$$

we call φ a homomorphism between $\mathbf{A}$ and $\mathbf{B}$.

Definition 11. If φ is a bijective homomorphism from $\mathbf{A}$ to $\mathbf{B}$, then we say φ is an *isomorphism*, and that $\mathbf{A}$ is *isomorphic* to $\mathbf{B}$, written $\mathbf{A} \cong \mathbf{B}$.

Theorem 3. *The kernel of a homomorphism is a congruence.*

Proof. Let $\ker(\varphi)$ be the kernel of a homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$. Then $\ker(\varphi)$ is an equivalence relation by Theorem 1 so it suffices to show that the kernel is a congruence on $\mathbf{A}$. Let $\langle a_i, b_i \rangle \in \ker(\varphi)$, for $1 \leq i \leq n$. Then

$$\begin{aligned}\varphi\left(f^A(a_1, \ldots, a_n)\right) &= f^A(\varphi(a_1), \ldots, \varphi(a_n)) \\ &= f^A(\varphi(b_1), \ldots, \varphi(b_n)) \\ &= \varphi\left(f^A(b_1, \ldots, b_n)\right).\end{aligned}$$

□

Theorem 4. *If θ is a congruence on A, then the canonical map $a \mapsto a/\theta$ is a homomorphism. We call this the* ***canonical homomorphism****.*

Proof. Let $\varphi : A \to A/\theta$ be the canonical map. Then φ is a homomorphism if for any n-ary operation f and elements $a_1, \ldots, a_n \in A$, $\varphi(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{A}/\theta}(\varphi(a_1), \ldots, \varphi(a_n))$. But $\varphi(a) = a/\theta$, so our equation becomes

$$f^{\mathbf{A}}(a_1, \ldots, a_n)/\theta = f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta),$$

which is precisely how the operations in the quotient algebra are defined (definition 9), thus when θ is a congruence, the canonical map on A by θ is a homomorphism. □

We can now state the Isomorphism theorems

Theorem 5. *(The First Isomorphism Theorem) If $\psi : \mathbf{A} \to \mathbf{B}$ is a homomorphism with $K = \ker(\psi)$, then K is a congruence on A. Let $\phi : \mathbf{A} \to \mathbf{A}/\ker(\psi)$ be the canonical homomorphism. Then there exists a unique isomorphism $\eta : \mathbf{A}/\ker(\psi) \to \psi(\mathbf{A})$ such that $\psi = \eta \circ \phi$.*

Proof. By Theorem 2 we know that η defined to be $a/K \mapsto \psi(a)$ is well defined and is a bijection from $\mathbf{A}/\ker(\psi)$ to $\psi(\mathbf{A})$. It remains to show that when ψ is a homomorphism, η is an isomorphism. Since $\mathbf{A}$ and $\mathbf{A}/K$ have the same signature, and $\mathbf{A}$ and $\mathbf{B}$ have the same signature, $\mathbf{A}/K$ and $\mathbf{B}$ have the same signature. Grab any n-ary function symbol f and elements $a_1/K, a_2/K, \ldots a_n/K$ from A/K. Then

$$\begin{aligned}
\eta\left(f^{A/K}(a_1/K, \ldots, a_n/K)\right) &= \eta\left(f^A(a_1, \ldots, a_n)/K\right) && \text{(since } A/K \text{ is a quotient algebra)} \\
&= \psi\left(f^A(a_1, \ldots, a_n)\right) && \text{(by definition of } \eta\text{)} \\
&= f^B(\psi(a_1), \ldots, \psi(a_n)) && \text{(since } \psi \text{ is a homomorphism)} \\
&= f^B(\eta(a_1/K), \ldots, \eta(a_n/K)) && \text{(by definition of } \eta\text{)}
\end{aligned}$$

Thus $\eta(f^{A/K}(a_1/K, \ldots, a_n/K) = f^B(\eta(a_1/K), \ldots, \eta(a_n/K))$, which shows that η is an isomorphism. $\square$

3.3 Subalgebras

Definition 12. (Subalgebra) Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras with the same signature $\mathscr{F}$. We say $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$, and for each $f \in \mathscr{F}$, the restriction of $f^{\mathbf{A}}$ to B is the same as $f^{\mathbf{B}}$. In other words, $\mathbf{B}$ "inherits" its operations from $\mathbf{A}$.

Definition 13. (Subuniverse) A *Subsuniverse* of an algebra $\mathbf{A}$ is a subset B of A such that B is closed under the operations of A. So the universe of any subalgebra is a subuniverse.

Note. This definition of a subalgebra is extremely general. In particular, we do not require that any inherited operation $f^{\mathbf{B}}$ satisfy any of the conditions that $f^{\mathbf{A}}$ may have been required to satisfy.

In an abstract algebra class, groups are usually introduced as structures with a single operation, and inverses and the identity element are "byproducts" of the group axioms. So why the additional complexity? One motivation comes from our definition of substructures. The following example motivates our need to define constant and unary functions even if they follow from the axioms of a given theory.

Example 4. (Counterexample) Suppose we did not require the function symbols $^{-1}$ or $\mathbf{1}$ in our definition of the signature for groups. Let $\mathbf{A} = \langle \mathbb{Z}, +^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle \mathbb{Z}^+, +^{\mathbf{B}} \rangle$. Then $\mathbf{A}$ and $\mathbf{B}$ are groups, and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, and $\mathbb{Z}^+$ is a subuniverse of $\mathbb{Z}$. But $\mathbf{B}$ would *not* be a subgroup of $\mathbf{A}$ since it is not a group at all. However, if $\mathbf{A} = \langle \mathbb{Z}, +^{\mathbf{A}}, ^{-1\mathbf{A}}, \mathbf{1}^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle \mathbb{Z}^+, +^{\mathbf{B}}, ^{-1\mathbf{B}}, \mathbf{1}^{\mathbf{B}} \rangle$, then B is a subuniverse of $\mathbf{A}$, but $\mathbf{B}$ is no longer a subalgebra.

3.4 Lattices

Although lattices are an algebra, they appear very naturally in universal algebra when studying algebras in general.

Definition 14. (Lattice) A Lattice is an algebra with universe L and a signature consisting of two binary operations $\wedge$ and $\vee$ called meet and join satisfying

$$\begin{aligned}
&a \vee b = b \vee a && a \wedge b = b \wedge a \\
&a \vee (b \vee c) = (a \vee b) \vee c && a \wedge (b \wedge c) = (a \wedge b) \wedge c \\
&a \vee a = a && a \wedge a = a \\
&a \vee (a \wedge b) = a && a \wedge (a \vee b) = a
\end{aligned}$$

for all $a, b, c \in L$.

Definition 15. (Complete Lattice) A Lattice is called a *complete lattice* if $\bigvee S$ and $\bigwedge S$ are defined for every subset S of L. In other words, for any subset $S \subseteq L$, $\exists u \forall s\ (s \vee u = u)$, and $\exists v \forall s\ (s \wedge v = v)$.

Example 5. Consider the integers $\mathbb{Z}$. Then $L = \langle \mathbb{Z}, \wedge, \vee \rangle$ is a lattice with $m \vee n = \max\{m, n\}$, and $m \wedge n = \min\{m, n\}$. However the the subset $\mathbb{Z}^+$ has no upper bound in $\mathbb{Z}$. In other words $\bigvee \mathbb{Z}^+$ is not defined since there are no elements u and v in $\mathbb{Z}$ such that $n \vee u = u$ and $n \wedge v = v$ for every $n \in \mathbb{Z}^+$. So L is not a complete lattice. But we could extend our lattice by adding the two element ∞ and $-\infty$ such that $n \vee \infty = \infty$, $n \wedge -\infty = -\infty$, and $n \wedge \infty = n$, $a \vee -\infty = n$ for all $n \in \mathbb{Z}$.

In the previous example (Example 5) we were able to turn our lattice into a complete lattice by "bounding" our lattice.

Definition 16. A Lattice is a called a *bounded lattice* if there exist elements I and O called the top and bottom such that for all $a \in L$

$$\begin{aligned} a \vee I &= I & \qquad a \wedge I &= a \\ a \vee O &= a & \qquad a \wedge O &= O. \end{aligned}$$

Note. Any complete lattice is a bounded lattice.

Definition 17. An element a of a lattice is called *compact* if whenever $a \vee \bigvee X = \bigvee X$, there is a finite subset $\{x_1, \ldots, x_k\}$ of X such that $a \vee x_1 \vee \ldots \vee x_k = x_1 \vee \ldots \vee x_k$.

Note. The term "compact" comes from the notion of compactness from topology in which a set is compact if every open covering has a finite covering.

Definition 18. A lattice is called *algebraic* if it is a complete lattice and every element can be written as the join of compact elements.

Theorem 6. *The set of all congruences on an algebra* $Con(\mathbf{A})$ *forms a complete lattice with* $\vee = \cup$ *and* $\wedge = \cap$. *The top is* A^2 *and the bottom is the diagonal relation* $\{\langle a, a \rangle \mid a \in A\}$, *or equivalently, then identity function on* A.

Definition 19. A congruence θ on an algebra $\mathbf{A}$ is called *fully invariant* if for any endomorphism $\sigma : \mathbf{A} \to \mathbf{A}$,

$$\langle a, b \rangle \in \theta \Rightarrow \langle \sigma(a), \sigma(b) \rangle \in \theta.$$

Theorem 7. *The set of all fully invariant congruences on an algebra form an algebraic sublattice of* $Con(\mathbf{A})$.

Theorem 8. *The set of all subalgebras of an algebra forms an algebraic lattice.*

Theorem 9. *(Birkhoff-Frink)*
Every algebraic lattice is isomorphic to the lattice of subalgebras for some algebra.

Theorem 10. *(Grätzer-Shchmidt)*
Every algebraic lattice is isomorphic to the lattice of congruences on some algebra.

4 First-order Logic

4.1 Relational Structures

So far in this paper, we have been careful to discuss lattices in purely in terms of the two operations $\vee$ and $\wedge$. However, when students are first introduced to the concepts of lattices and Boolean algebras, many textbooks will first introduce partially ordered set, or posets. A poset consists only of a set P, and a relation on P, usually denoted $\preccurlyeq$, satisfying some properties. While we can prove that there is a correspondence between certain kinds of posets and lattices, a poset is actually *not* an algebra, as it has no operations. Structures like posets are instead called *relational structures*.

Since the definition is so similar to signatures (Definition 6), we will be less formal in defining relational structures.

Definition 20. (Relational structure) A *relational structure* with signature $\mathscr{R}$ is a pair $\langle A, R\rangle$ where $\mathscr{R}$ is a set of relation symbols, with corresponding relations on A in R.

Example 6. A poset $\mathbf{P} = \langle P, \preccurlyeq^{\mathbf{P}}\rangle$ is a relational structure with signature $\{\preccurlyeq\}$.

4.2 First-order Languages and Structures

First-order languages in this context can be viewed as generalizing signatures defined earlier (Definition 6) to encompass both algebras and relational structures. A signature or type for algebras can also be called a *language of algebras*.

Definition 21. (First-order Language) A first order language $\mathscr{L}$ consists of a set of function symbols, a set of relation symbols, a quantifier symbol '$\forall$', as well as the following "logical" symbols

- parentheses: '(', ')'
- connectives: '$\Rightarrow$' (implication), '$\neg$' (negation)
- variables: 'v_1', 'v_2', ...
- equality: '$=$'

Note. We have not included the quantifier '$\exists$', or several other connectives because we can define these in terms of the symbols we have included. Define $\exists x\ p(x)$ to mean$\neg\forall x\ \neg p(x)$. Other connectives can also be defined in terms of $\Rightarrow$ and $\neg$. Define $p \,\&\, q$ as $\neg(p \Rightarrow \neg q)$.

Definition 22. (First-order Structure) A first order structure with a language $\mathscr{L}$, or simply an $\mathscr{L}$-structure, consists of a universe, a set of functions, and a set of relations which correspond to the functions and relations in $\mathscr{L}$.

Example 7. An *ordered field* is an $\mathscr{L}$-structure with a signature $\{+, \cdot, -, ^{-1}, \mathbf{0}, \mathbf{1}, <\}$. For example the real numbers as a field and with their usual ordering is the real ordered field. Notice that this structure is neither an algebra nor a relational structure.

Up to this point, we have not discussed the usual constraints placed on functions or relations. For example, a group is not just an algebra with the signature for groups. We further require that the operations satisfy the group axioms. In model theory we have a more rigorous definition of what it means to *satisfy* a set of axioms.

An important concept in logic is that of a *well-formed formula*. For a formal description of well-formed formulas in first-order logic, see Chapter 2 of [3].

Definition 23. Informally, a *well-formed formula*, or wff for short, is any (mathematically) "meaningful" sequence of symbols that can be built from the symbols in our language. We call a wff with no free variables a *sentence*.

Example 8. Consider the following sequences of symbols in a first-order language

$$)f_1\neg\forall \tag{1}$$
$$\forall v_2\,(f_1(v_1,v_2) \Rightarrow f_1(v_2,v_1)) \tag{2}$$
$$\forall v_1\forall v_2\,(f_1(v_1,v_2) \Rightarrow f_1(v_2,v_1)) \tag{3}$$

(1) is not a well-formed formula, (2) is a wff which is *not* a sentence since the variable v_1 is free, and (3) is a sentence.

We will now give a very informal definition of what it means to *satisfy* a sentence or set of sentences.

Definition 24. For a given language $\mathscr{L}$, a sentence σ of $\mathscr{L}$, and an $\mathscr{L}$-structure $\mathbf{A}$, we say that $\mathbf{A}$ *satisfies* σ if when we "translate" σ into the corresponding functions and relations of $\mathbf{A}$, σ is true.

Definition 25. (Models) An $\mathscr{L}$-structure $\mathbf{A}$ is a *model* of a sentence σ of $\mathscr{L}$ if $\mathbf{A}$ satisfies σ.

Note. We also say that $\mathbf{A}$ is a model of a set of sentences Σ if $\mathbf{A}$ satsifies every sentence in Σ.

Example 9. Let $\mathscr{L}$ be the first-order language with only a single relation symbol R, and let $\mathbf{G}$ be the $\mathscr{L}$-structure with universe $G = \{a, b, c\}$, and $R^{\mathbf{G}} = \{\langle a,b\rangle, \langle a,c\rangle, \langle b,c\rangle, \langle c,c\rangle\}$. We can visualize $\mathbf{G}$ as the following graph.

$$\begin{tikzcd} a \arrow[rr] \arrow[dr] & & b \arrow[dl] \\ & c \arrow[loop below] & \end{tikzcd}$$

Let σ be the sentence
$$\exists v_1\,\forall v_2\,(v_2 R v_1).$$

In English, σ says that there exists an element which every element points to. In fact $\mathbf{G}$ satisfies σ, since there in fact *is* an element like this, namely c. Thus $\mathbf{G}$ is a model of the sentence σ.

Definition 26. (Logical consequence) A sentence σ is a *logical consequence* of a set of sentences Γ if whenever $\mathbf{A}$ is a model of Γ, $\mathbf{A}$ is a model of σ. We write $\Gamma \vDash \sigma$ to mean σ is a logical consequence of Γ, or equivalently, Γ *logically implies* σ.

Definition 27. (Theory) A *Theory* is a set of sentences which is closed under logical implication. In other words T is a theory if
$$T \vDash \sigma \Rightarrow \sigma \in T$$
.

Definition 28. A theory T is said to be *finitely axiomatizable* if there exists a finite subset Σ of T such that every sentence in T is a logical consequence of Σ.

Example 10. The theory of groups is a first-order finitely axiomatizable theory.

Sources

[1] G. Birkhoff. "On the Structure of Abstract Algebras". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 31 (Oct. 1935), pp. 433–454. DOI: 10.1017/S0305004100013463. URL: http://math.hawaii.edu/~ralph/Classes/619/birkhoff1935.pdf.

[2] S. Burris and H.P. Sankappanavar. *A Course in Universal Algebra*. Graduate Texts in Mathematics. Springer-Verlag, 1981–2012. URL: http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html.

[3] H.B. Enderton. *A Mathematical Introduction to Logic*. Elsevier Science, 2001. ISBN: 9780080496467.

[4] G.A. Grätzer. *Universal algebra*. University series in higher mathematics. Van Nostrand, 1968.

[5] P. Jipsen. *Tutorial on Universal Algebra*. presentation from 2009. URL: http://mathcs.chapman.edu/~jipsen/talks/BLAST2009/JipsenUAtutorial1pp.pdf.

[6] T. W. Judson. *Abstract Algebra: Theory and Applications*. 2019. URL: http://abstract.pugetsound.edu/aata.

[7] J.B. Nation. *Notes on Lattice Theory*. URL: http://www.math.hawaii.edu/~jb/books.html.

[8] Eric W. Weisstein. *Universal Algebra. From MathWorld—A Wolfram Web Resource*. visited on March 23rd, 2019. URL: http://mathworld.wolfram.com/UniversalAlgebra.html.