Math 290
Exam 3
Chapter M

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Use the inverse of a matrix to find the solution set for the linear system $A\mathbf{x} = \mathbf{b}$. No credit will be given for other methods, so be sure to explain fully. (15 points)

$$A = \begin{bmatrix} -2 & 4 & -3 \\ 4 & -7 & 5 \\ -3 & 6 & -5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$[A|I_3] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & -1 \\ 0 & 1 & 0 & 5 & 1 & -2 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right]$$

A nonsingular NMRRI

A^{-1} by CINM + OSIS

$$A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \underset{\sim}{x} = A^{-1}\underset{\sim}{b}$$

Theorem SNCM

$$= \begin{bmatrix} 5 & 2 & -1 \\ 5 & 1 & -2 \\ 3 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

2. Find a non-zero vector $\mathbf{c}$ so that $A\mathbf{x} = \mathbf{c}$ is a consistent system, and a vector $\mathbf{d}$ so that $A\mathbf{x} = \mathbf{d}$ is an inconsistent system. In each case, *without actually attempting to solve a system of equations*, explain clearly how you know your answer is correct. So, for example, "guess and check" is not an explanation. Hint: a particular version of the column space of A could prove useful. (20 points)

$$A = \begin{bmatrix} 1 & 4 & -1 & 2 \\ -1 & -3 & 1 & -2 \\ 1 & 2 & -1 & 2 \end{bmatrix}$$

$$C(A) = R(A^t) \qquad A^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{So } C(A) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\} \right\rangle$$

By Theorem CSCS, any linear combo of these two vectors will give a $\underset{\sim}{c}$.

$$\underset{\sim}{c} = 4\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

← Ha! column 2 of A!

Any vector in C(A) that looks like $\begin{bmatrix} 4 \\ -3 \\ \square \end{bmatrix}$ must have a 2 in slot 3.

So $\underset{\sim}{d} = \begin{bmatrix} 4 \\ -3 \\ 899 \end{bmatrix} \notin C(A)$ and will create an inconsistent system.

3. In each part of this question, find a set S of column vectors so that the span of S equals the column space of the matrix A (that is, $\mathcal{C}(A) = \langle S \rangle$) and S satisfies the additional requirements given in each part. Provide clear explanations on why your answer meets all the requirements. (25 points)

$$A = \begin{bmatrix} 87 & -348 & -124 & -11 & -633 & 50 \\ -35 & 140 & 50 & 4 & 255 & -20 \\ -12 & 48 & 17 & 2 & 87 & -7 \\ -7 & 28 & 10 & 1 & 51 & -4 \end{bmatrix}$$

$$[A \mid I] \rightarrow \left[\begin{array}{cccccc|cccc} 1 & -4 & 0 & 0 & -3 & 2 & 0 & -3 & -10 & 32 \\ 0 & 0 & 1 & 0 & 3 & 1 & 0 & -2 & -7 & 22 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right]$$

(a) S is linearly independent and is a subset of the set of columns of A.

Pivot columns of row-reduced A = $D = \{1, 3, 4\}$.

S = columns 1,3,4 of A = $\left\{ \begin{bmatrix} 87 \\ -35 \\ -12 \\ -7 \end{bmatrix}, \begin{bmatrix} -124 \\ 50 \\ 17 \\ 10 \end{bmatrix}, \begin{bmatrix} -11 \\ 4 \\ 2 \\ 1 \end{bmatrix} \right\}$

Theorem BCS says S spans & is linearly independent.

(b) S is linearly independent and is computed using results about row spaces.

$C(A) = R(A^t)$ = nonzero rows of row-reduced A^t

Theorem CSRST

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Theorem BRS says S spans $R(A^t)$ & is lin. ind.

(c) S is linearly independent and is computed using the matrix L from the extended echelon form of A.

$L = [1 \; 2 \; 2 \; -1]$ and is already in RREF

$C(A) = N(L) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

Theorem FS

"pattern of zeros and ones" in slots 2, 3, 4

Theorem BNS gives the span & linear independence requirements.

4. Using the same matrix A from the previous question, find a set S of column vectors so that the span of S equals the row space of the matrix A (that is, $\mathcal{R}(A) = \langle S \rangle$) and S is a linearly independent set. Provide a clear explanation on why your answer meets all the requirements. (10 points)

Theorem BRS says we only need the nonzero rows of the row-reduced version of A (at the top of the page).

$S = \left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

5. Suppose that A is an $m \times n$ coefficient matrix of a homogeneous system where columns i and j are identical ($1 \leq i < j \leq n$). Without analyzing the reduced row-echelon form of the coefficient matrix, prove that the system has infinitely many solutions. (15 points)

Write $A = [\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_n]$. Then define $\mathbf{x}$ by

$$[\mathbf{x}]_k = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}. \quad \text{In other words,} \quad \mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{slot } i \\ \\ \\ \\ \leftarrow \text{slot } j \\ \\ \\ \\ \end{matrix}$$

Then

$$A\mathbf{x} = 0\mathbf{A}_1 + \cdots + 1\mathbf{A}_i + \cdots - 1\mathbf{A}_j + \cdots + 0\mathbf{A}_n = \mathbf{0}$$

So $\mathbf{x}$ is a non-zero solution to $A\mathbf{x} = \mathbf{0}$. And $A\mathbf{0} = \mathbf{0}$.

So the homogeneous system has two solutions & hence must have infinitely many solutions.

6. Suppose that A is an $m \times n$ matrix, and B and C are both $n \times p$ matrices. Prove the conclusion of Theorem MMDAA: $A(B + C) = AB + AC$. (15 points)

This is a matrix equality between $m \times p$ matrices. So for $1 \leq i \leq m$, $1 \leq j \leq p$

$$\begin{aligned}
[A(B+C)]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [B+C]_{kj} \\
&= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) \\
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj} \\
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj} \\
&= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}
\end{aligned}$$

So by Defn ME we have $A(B+C) = AB + AC$

See proof of MMDAA.