Math 491, Thursday, April 30, Galois Theory, Chapter 23

Friday - Problem Session 23
Sage 23
(Presentations
Housekeeping) < Sun 11:59 PM
Presentations → RAB
Written (final)

Mon - Exam 4 22/23

Tue - Presentations (3x, 8 AM start)

Tue - Final Exam
10 AM - 1 PM

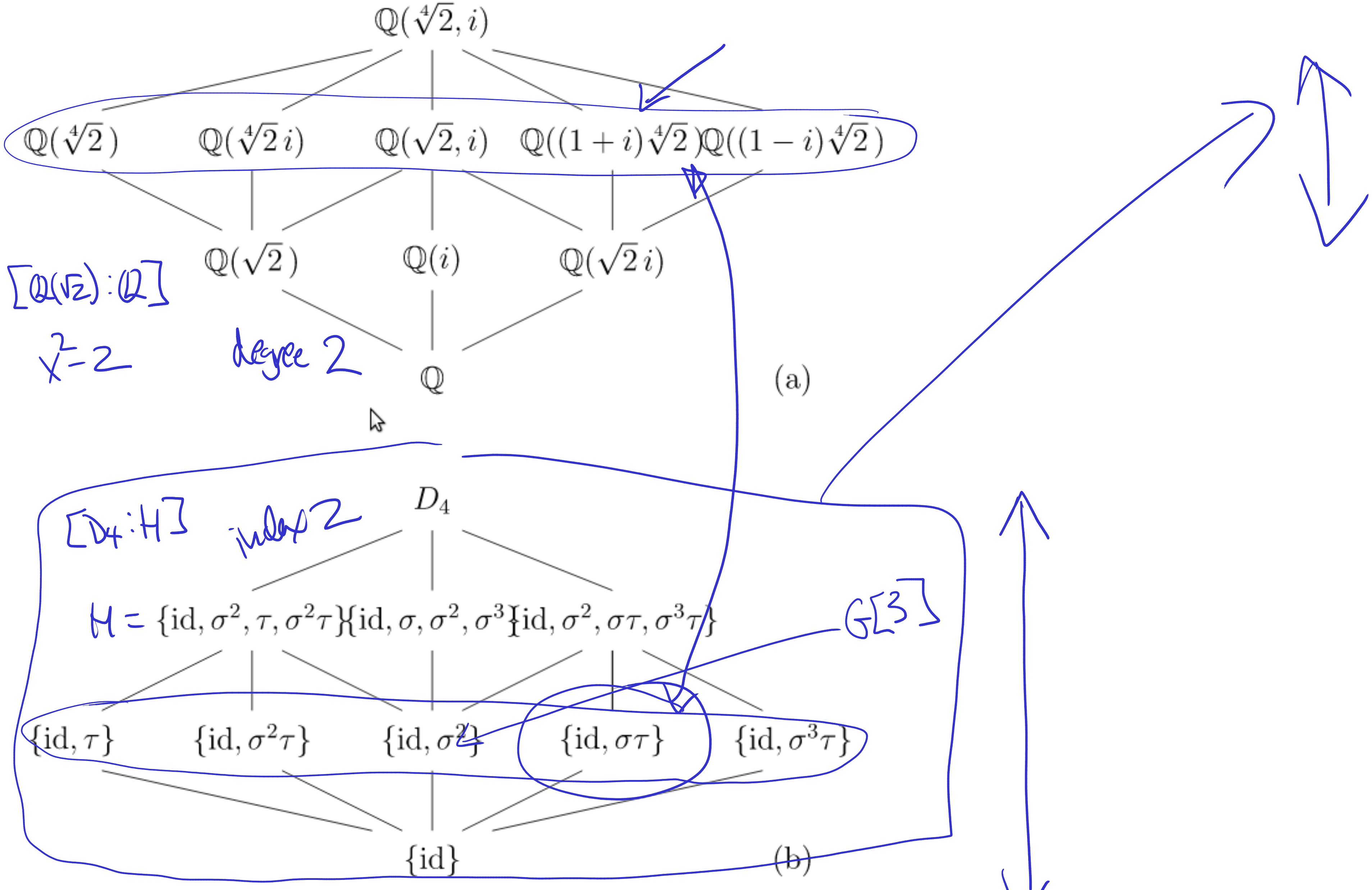


Figure 23.25. Galois group of $x^4 - 2$

Insolvability of Quintic

<u>Defn</u> A tower of extensions $F(a_1, a_2, \dots a_k)$ is an <u>extension by radicals</u> if $a_i^{n_i} \in F(a_1, a_2, \dots, a_{i-1})$ for some $n_i \in \mathbb{Z}^+$, for each i.

$$\begin{array}{c} F(a_1, \dots, a_n) \\ | \\ \vdots \\ | \\ F(a_1, \dots, a_i) \\ | \\ F(a_1, \dots, a_{i-1}) \\ | \\ F \end{array}$$

$x^{n_i} - f_i$, $f_i \in F(a_1, \dots, a_{i-1})$

$a_i = \sqrt[n_i]{f_i}$

<u>Defn</u> A polynomial is solvable by radicals if its splitting field is contained in an extension by radicals.

A "formula" for roots of a poly is a combination of field operations & "taking" radicals

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solvable groups

$$\{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_k = G$$

- "subnormal" $H_i \triangleleft H_{i+1}$
- "composition" factors H_{i+1}/H_i simple (no normal subgroups)
- "solvable" H_{i+1}/H_i abelian

"Most" small groups are solvable
- S_5 smallest non-solvable group
- $PSL(2,7)$ has order 168, simple (non-abelian) non-solvable

Feit-Thompson Theorem: Every group of odd order is solvable.

Theorem The splitting field of $x^n - a$ over a field of characteristic zero has a solvable Galois group.

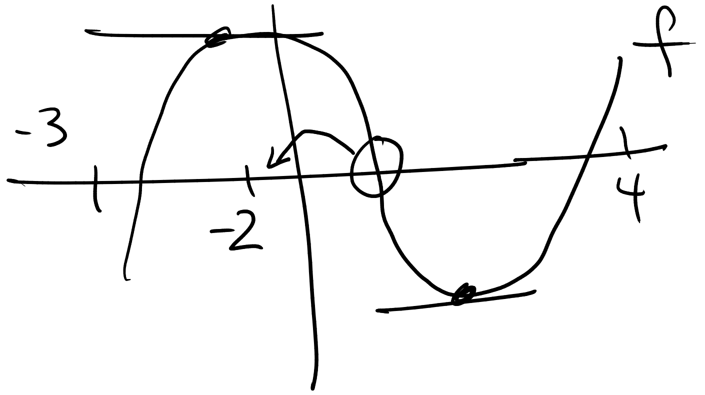
Theorem $f(x) \in F[x]$, $\mathrm{char}(F) = 0$

If $f(x)$ solvable by radicals $\Rightarrow$ Galois group of f is solvable.

Ex $f(x) = x^5 - 6x^3 - 27x - 3$ not solvable by radicals

$\in \mathbb{Q}[x]$

- Eisenstein $p=3$ $\to$ irreducible
- $f'(x)$ has exactly two roots, at most a max & min
- sign changes at $x = -3, -2, 0, 4$
  IVT $\Rightarrow$ exactly three real roots
- So 5 roots, must have two complex roots
- Permute roots : | complex conjugation $\leftarrow$ field automorphism
  switch two complex, fix three real] transposition



- Irreducible polynomial $\Rightarrow$ degree 5 extension by root a

$$\begin{array}{c} \mathbb{Q}(a) \\ 5 \,| \\ \mathbb{Q} \end{array} \Rightarrow 5 \mid \text{order of Galois group}$$

$\Rightarrow$ element of order 5 in Galois group

5-cycle in permutation group

τ

$\sigma \quad \tau \Rightarrow$

- $\langle \underline{(12)}, (\underline{12}345) \rangle = S_5$

WLOG

$\sigma^2 \tau \sigma$

- Galois Group S_5
- Composition series $\{e\} \triangleleft A_5 \triangleleft S_5$

factor $A_5/\{e\} \cong A_5$ not abelian

- So S_5 not <u>solvable</u>!