Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2021
Chapter E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers, unless explicitly suggested in the problem's statement. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Consider the matrix A below. (35 points)

   (a) Compute some potential eigenvalues of A. Eventually, you will want to do this in a way that you find all of the eigenvalues of A, so be thorough. You may use Sage to factor polynomials.

$$A = \begin{bmatrix} 5 & 1 & -5 & 2 \\ 3 & 3 & -4 & 3 \\ 3 & 0 & -2 & 2 \\ -3 & -1 & 4 & -1 \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad [A^0\mathbf{e}_1 | A^1\mathbf{e}_1 | A^2\mathbf{e}_1 | A^3\mathbf{e}_1 | A^4\mathbf{e}_1] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

many other choices

A "short" solution in the null space sets $x_3 = 1$, $x_4 = 0$, $x_5 = 0$ for a RLD

$-2A^0\mathbf{e}_1 + (-1)A^1\mathbf{e}_1 + 1A^2\mathbf{e}_1 = \mathbf{0}$, so a polynomial is

$-2 - x + x^2 = (x-2)(x+1)$ w/ roots as potential eigenvalues: $\lambda = 2$, $\lambda = -1$

   (b) For each of your potential eigenvalues compute the traditional eigenspace. Explain how this allows you to now be certain which of your results in part (a) are definitely eigenvalues of A.

Theorem ESM λ eigenvalue $\Leftrightarrow$ $A - \lambda I$ singular

$\lambda = 2$

$$A - 2I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{singular}$$

$$\mathcal{E}_A(2) = \left\langle \left\{ \begin{bmatrix} -2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\lambda = -1$

$$A - (-1)I_4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{singular}$$

$$\mathcal{E}_A(-1) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

   (c) For each eigenvalue, compute a generalized eigenspace and an algebraic multiplicity. Explain how you can now be certain that you have computed every eigenvalue of A.

theorem NEM $\sum \alpha_A(\lambda_i) = n$

$\lambda = 2$

$$(A - 2I_4)^4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{G}_A(2) = \left\langle \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \quad \alpha_A(2) = 3$$

$\lambda = -1$

$$(A - (-1)I_4)^4 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(same as above)}$$

$$\mathcal{G}_A(-1) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle \quad \alpha_A(-1) = 1$$

$3 + 1 = 4$! $\Rightarrow$ all eigenvalues accounted for

   (d) What is the characteristic polynomial of A?

Definition CP

$$p_A(x) = (x-2)^3 (x-(-1))^1$$

2. Consider the matrix B below. (25 points)

(a) Use Sage's `.eigenvalues()` method to compute the eigenvalues of B. Then, by simply row-reducing the right matrices, determine the geometric and algebraic multiplicity of each eigenvalue.

$$B = \begin{bmatrix} 8 & 20 & -10 & 20 \\ -5 & -2 & -5 & -20 \\ -5 & 0 & -7 & -20 \\ 0 & -5 & 5 & 3 \end{bmatrix}$$

From B.eigenvalues(), $\lambda = 3$, $\lambda = -2$

Row-reduce indicated matrices, count zero rows (or non-pivots, identical for square matrices) to get nullities, which are dimensions of eigenspaces

$B - 3I_4 \Rightarrow \gamma_B(3) = 2$] equal

$(B-3I_4)^4 \Rightarrow \alpha_B(3) = 2$

$B - (-2)I_4 \Rightarrow \gamma_B(-2) = 2$] equal

$(B-(-2)I_4)^4 \Rightarrow \alpha_B(-2) = 2$

(b) Explain how you now know that B is diagonalizable.

Theorem ~~DM~~FE ("full eigenspaces") implies B is diagonalizable since algebraic multiplicity and geometric multiplicity are equal <u>for each</u> eigenvalue.

(c) Again, just by row-reducing the right matrices, find a nonsingular matrix S which will diagonalize B. Give the resulting diagonal matrix D.

Combine two bases for the traditional eigenspaces of B

$B - 3I_4 \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{E}_B(3) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

$B + 2I_4 \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{E}_B(2) = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 6 \\ 1 \end{bmatrix} \right\} \right\rangle$

$\Rightarrow S = \begin{bmatrix} -2 & -4 & 1 & -4 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Theorem DC guarantees that

$S^{-1}BS = \begin{bmatrix} 3 & & & 0 \\ & 3 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$

3. Suppose that 0 is an eigenvalue of A. Prove that A is singular. (10 points)

(1) Let $\underset{\sim}{x}$ be an eigenvector of A for $\lambda = 0$. Then $\underset{\sim}{x} \neq 0$, and $A\underset{\sim}{x} = 0\underset{\sim}{x} = \underset{\sim}{0}$ so A is singular

OR

(2) $\lambda = 0$ is eigenvalue $\Leftrightarrow$ $A - 0I_n$ singular

<u>this</u> is just A

4. Prove the transitive property of an equivalence relation for similarity. That is, suppose A and B are similar matrices, and B and C are similar matrices. Conclude that A and C are similar matrices. (15 points)

There exist S & T (different!) so that $AS = SB$, $BT = TC$.

Consider $A(ST)$

$$A(ST) = (AS)T = (SB)T = S(BT) = S(TC) = (ST)C$$

This says A is similar to C, via ST.

Note: S nonsingular (invertible), T nonsingular (invertible) $\Rightarrow$ ST nonsingular (invertible) Theorem SOX & Shoes!

5. Suppose that A is an $m \times n$ matrix. Prove that the column space of A, $\mathcal{C}(A)$, is an A-invariant subspace of $\mathbb{C}^m$. (15 points)

Grab $\underset{\sim}{x} \in C(A)$. Consider $A\underset{\sim}{x}$.

We know a matrix-vector product is a linear combination of the columns of the matrix. So $A\underset{\sim}{x}$ is a linear combination of the columns of A, hence $A\underset{\sim}{x} \in C(A)$.

This establishes that $C(A)$ is A-invariant.