Math 290 Name: Dr. Beezer
Exam 6 Spring 2021
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers, unless explicitly suggested in the problem's statement. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Consider the function $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ below. (35 points)

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a - b \\ a + 3b \end{bmatrix}$$

   (a) Prove that T is a linear transformation.

   (b) Is T injective? Why or why not?

2. Consider the linear transformation $T\colon S_{22} \to P_2$ defined below, where S_{22} is the vector space of 2×2 symmetric matrices, and P_2 is the vector space of polynomials of degree at most 2. (35 points)

$$T\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (3a + b + 5c) + (2a + b + 4c)\,x + (-3a + 4b + 5c)\,x^2$$

(a) Compute the kernel of T, $\mathcal{K}(T)$.

(b) Compute the range of T, $\mathcal{R}(T)$.

(c) The rank and nullity of T obey a basic relationship. Say what this relationship is, and verify it for T.

(d) Compute the preimage of $2 + x - 7x^2$, $T^{-1}\left(2 + x - 7x^2\right)$.

3. For the linear transformation $R\colon M_{22} \to \mathbb{C}^3$ find a specific element of the codomain with an empty pre-image, demonstrating that R is not surjective. (M_{22} is the vector space of 2×2 matrices.) (15 points)

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2c + 2d \\ a - b + 3c + d \\ b - c + d \end{bmatrix}$$

4. Suppose that $S\colon U \to V$ and $T\colon V \to W$ are linear transformations and each is injective. Prove that their composition, $T \circ S$ is invertible. (15 points)