Math 181 Friday, March 26 Section 10.6

<u>Ex</u> $h(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ interval of convergence?

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1} x^{n+1}}{\frac{1}{n} x^n} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right) |x|$$

$$= |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$

absolute convergence if $L < 1 \longrightarrow |x| < 1 \longrightarrow -1 < x < 1$

check endpoints $x = 1$ $h(1) = \sum_{n=1}^{\infty} \frac{1}{n} 1^n = \sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic series p-series, $p=1$ diverges

$x = -1$ $h(-1) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n$ converges (conditionally) by Alternating Series test

Interval: $-1 \le x < 1$

Ex $\ell(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad (0! = 1)$ Interval of convergence?

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} |x| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1 \text{ for } \underline{\underline{\text{all}}}\ x$$

Interval of Convergence: $(-\infty, \infty)$ // $\ell(x)$ is defined for $\underline{\text{all}}$ x.

Differentiate $\ell(x)$???

$$\frac{d}{dx}\ell(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \underline{\underline{n}} x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = \sum_{K=0}^{\infty} \frac{1}{K!} x^K = \ell(x)$$

Just like e^x!!!! $\ell(x)$ converges to e^x for all x.

For example

$$e^1 = \ell(1) = \sum_{n=0}^{\infty} \frac{1}{n!} 1^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots\cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$$

Ex $\quad r(x) = \sum_{n=0}^{\infty} \frac{n!}{2^n} x^n \quad$ interval of convergence

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!/2^{n+1} \quad x^{n+1}}{n!/2^n \quad x^n} \right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \frac{(n+1)!}{n!} |x|$$

$$= \lim_{n\to\infty} \frac{1}{2} \, (n+1) \, |x| \;\; \boxed{\cancel{= \frac{1}{2}|x| \underbrace{\lim_{n\to\infty} n+1}_{DNE}}} = \begin{cases} \lim\limits_{n\to\infty} 0 & \text{if } x=0 \\ \lim\limits_{n\to\infty} \frac{|x|(n+1)}{2} & \text{if } x \neq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x=0 \\ DNE & \text{if } x \neq 0 \end{cases}$$

$L < 1$ only if $x = 0$

$r(x)$ converges only for $x=0$ (and $r(0) = 0$)

Ex $\displaystyle k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Ratio Test, $L = 0$ for all x. Careful w/ $\dfrac{x^{2(n+1)}}{x^{2n}} = \dfrac{x^{2n+2}}{x^{2n}} = x^2$

$$k(x) = \underset{n=0}{1} - \underset{n=1}{\frac{x^2}{2}} + \underset{n=2}{\frac{x^4}{24}} - \frac{x^6}{720} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\cdots$$

Converges absolutely for all x, <u>to $\cos(x)$.</u> [take two derivatives]

Physicists: Replace $\cos(x)$ by $1 - \frac{x^2}{2}$ for <u>small</u> x.