Math 290 Friday, March 26 Section PEE

Thu IS
Fri Problems

Writing VS-2
PD.T60 Solution 1

Theorem EDELI

A matrix, $S = \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ eigenvectors of A for distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_p$

$\Rightarrow$ S is linearly independent

Proof Induction on p

Base Case $p=1$ $S = \{\underset{\sim}{x}_1\}$ $\underset{\sim}{x}_1$ - eigenvector $\Rightarrow \underset{\sim}{x}_1 \neq \underset{\sim}{0}$

so yes, S is linearly independent.

Induction Step Assume theorem true for sets with $p-1$ vectors

Want to prove the theorem for sets of p vectors

RLD on $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$

$$\underset{\sim}{0} = a_1 \underset{\sim}{x}_1 + a_2 \underset{\sim}{x}_2 + \cdots + a_p \underset{\sim}{x}_p$$

① $(A - \lambda_p I)\, \underset{\sim}{0} = \underset{\sim}{0}$

② $(A - \lambda_p I)(a_1 \underset{\sim}{x}_1 + a_2 \underset{\sim}{x}_2 + \cdots + a_p \underset{\sim}{x}_p)$

$$= a_1 (A - \lambda_p I) \underset{\sim}{x}_1 + a_2 (A - \lambda_p I) \underset{\sim}{x}_2 + \cdots + a_p (A - \lambda_p I) \underset{\sim}{x}_p$$

$$= a_1 (A\underset{\sim}{x}_1 - \lambda_p \underset{\sim}{x}_1) + a_2 (A\underset{\sim}{x}_2 - \lambda_p \underset{\sim}{x}_2) + \cdots + a_p (A\underset{\sim}{x}_p - \lambda_p \underset{\sim}{x}_p)$$

$$= a_1 (\lambda_1 \underset{\sim}{x}_1 - \lambda_p \underset{\sim}{x}_1) + a_2 (\lambda_2 \underset{\sim}{x}_2 - \lambda_p \underset{\sim}{x}_2) + \cdots + a_p \boxed{(\lambda_p \underset{\sim}{x}_p - \lambda_p \underset{\sim}{x}_p)}$$

$$= a_1 (\lambda_1 - \lambda_p) \underset{\sim}{x}_1 + a_2 (\lambda_2 - \lambda_p) \underset{\sim}{x}_2 + \cdots + a_{p-1} (\lambda_{p-1} - \lambda_p) \underset{\sim}{x}_{p-1}$$

② + ① $\Rightarrow$ $\underset{\sim}{0} = a_1 (\lambda_1 - \lambda_p) \underset{\sim}{x}_1 + \cdots + a_{p-1} (\lambda_{p-1} - \lambda_p) \underset{\sim}{x}_{p-1}$

RLD on $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_{p-1}\}$ $\leftarrow$ By induction, this set is linearly independent

So $a_1 \underbrace{(\lambda_1 - \lambda_p)}_{\text{nonzero}} = 0$, $a_2 \underbrace{(\lambda_2 - \lambda_p)}_{\text{nonzero}} = 0$, $\ldots$ $a_{p-1} \underbrace{(\lambda_{p-1} - \lambda_p)}_{\text{nonzero}} = 0$

So $a_1 = 0$, $a_2 = 0$, ..., $a_{p-1} = 0$

Original RLD is now $0\underset{\sim}{x}_1 + 0\underset{\sim}{x}_2 + \cdots + 0\underset{\sim}{x}_{p-1} + a_p \underset{\sim}{x}_p = \underset{\sim}{0}$

$a_p \underset{\sim}{x}_p = \underset{\sim}{0}$, $\underset{\sim}{x}_p \neq 0$

Theorem SMEZV $\Rightarrow a_p = 0$

So $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_p\}$ <u>is</u> a linearly independent set.

<u>Theorem</u> A is an $n \times n$ matrix. Then A has at most n distinct eigenvalues.

<u>Proof</u> If A has $n+1$ or more distinct eigenvalues,

a set of $n+1$ or more eigenvectors will be linearly independent

$\dim(\mathbb{C}^n) = n$, Theorem G says set is linearly dependent

Or MVSLD Theorem also can be used

Therem A Hermitian matrix (self-adjoint) $A = A^*$ then the eigenvalues of A are real numbers. $\mathbb{R}^n$

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$5 \cdot 0 = 3 \cdot 0$$

Proof Let $\underset{\sim}{x}$ be an eigenvector w/ eigenvalue λ.

$$\lambda \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = \langle \underset{\sim}{x}, \lambda \underset{\sim}{x} \rangle = \langle \underset{\sim}{x}, A\underset{\sim}{x} \rangle = \langle A^* \underset{\sim}{x}, \underset{\sim}{x} \rangle = \langle A\underset{\sim}{x}, \underset{\sim}{x} \rangle = \langle \lambda \underset{\sim}{x}, \underset{\sim}{x} \rangle$$

$$= \overline{\lambda} \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle$$

Notice that $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \neq 0$ because $\underset{\sim}{x} \neq \underset{\sim}{0}$, Theorem PIP

So $\lambda = \overline{\lambda}$ $\Rightarrow$ imaginary / complex part is zero

$\Rightarrow$ λ is real