Math 390 Friday, April 23 Determinants

Sun Nite 11:59 PM
Final Projects

Monday – Determinants
Tuesday – Problems
Thu, Fri – Prepare Presentations

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

<u>Theorem</u> Suppose $B$ is formed from $A$ by multiplying row $k$ by the scalar $\alpha$. Then $\det(B) = \alpha \det(A)$

<u>Proof</u>

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots (\alpha a_{k\sigma(k)}) \cdots a_{n\sigma(n)}$$

$$= \alpha \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \alpha \det(A)$$

<u>Corollary</u> If $A$ has a row of all zeros, then $\det(A) = 0$.

<u>Proof</u> Let $B$ be the matrix $A$, except the row of zeros is multiplied by zero. Note: $B = A$

$$\det(A) = \det(B) \underset{\nwarrow \text{ previous theorem}}{=} 0 \det(A) = 0$$

<u>Theorem</u> Suppose $B$ is the matrix obtained from $A$ by swapping rows $i$ & $j$. Then $\det(B) = -\det A$

<u>Proof</u> Let $\rho$ be permutation swaps $i$ & $j$, fixes everything else.

$$\rho = 1\ 2 \cdots j \cdots i \cdots n{-}1\ n \qquad i < j.$$

$$\det B = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, b_{1\sigma(1)}\, b_{2\sigma(2)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\boxed{\sigma} \in S_n} \operatorname{sign}(\sigma\rho)\, b_{1(\sigma\rho)(1)}\, b_{2(\sigma\rho)(2)} \cdots b_{n(\sigma\rho)(n)}$$

$$= \sum_{\sigma \in S_n} (-1)\, \text{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{i\,\sigma p(i)} \cdots b_{j\,\sigma p(j)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)\, \text{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{i\sigma(j)} \cdots b_{j\sigma(i)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)\, \text{sign}(\sigma)\, a_{1\sigma(1)} \cdots \underline{a_{j\sigma(j)}} \cdots \underline{a_{i\sigma(i)}} \cdots a_{n\sigma(n)}$$

$$= (-1) \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)}$$

$$= (-1) \det(A).$$

<u>Corollary</u> Suppose $A$ has two identical rows, then $\det(A) = 0$.

<u>Proof</u> Let $B$ be the matrix formed from $A$ by swapping the two rows.

$$\det(A) = \det(B) = -\det(A) \Rightarrow 2\det(A) = 0 \Rightarrow \det(A) = 0.$$

$A = B$ Previous theorem

**Theorem** Form matrix $B$ by adding a multiple of row $i$ to row $j$ (from $A$)

Then $\det(A) = \det(B)$.

**Proof**

$$\det B = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots (\alpha a_{i\sigma(j)} + a_{j\sigma(j)}) \cdots a_{n\sigma(n)}$$

$$= \alpha \sum_{\sigma \in S_n} \operatorname{sign}\sigma\, a_{1\sigma(1)} \cdots \underline{a_{i\sigma(j)}} \cdots a_{n\sigma(n)} + \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots \underbrace{a_{j\sigma(j)}}_{\text{slot } j} \cdots a_{n\sigma(n)}$$

matches $a_{i\sigma(i)}$

$$= 0 + \det(A)$$

determinant of a matrix where rows $i$ & $j$ are identical

# Axiomatic Definition of Determinant

Define a function of a matrix to the complex numbers, using the rows of the matrix (as vectors).

$D : M_{nn} \to \mathbb{C}$ $\qquad D(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_n)$

<u>Require</u>

1) $D(\underset{\sim}{r}_1, \dots, \alpha \underset{\sim}{r}_k, \dots, \underset{\sim}{r}_n) = \alpha D(\underset{\sim}{r}_1, \dots, \underset{\sim}{r}_n)$

2) $D(\underset{\sim}{r}_1, \dots, \underset{\sim}{r}_k + \underset{\sim}{r}, \dots, \underset{\sim}{r}_n) = D(\underset{\sim}{r}_1, \dots, \underset{\sim}{r}_k, \dots, \underset{\sim}{r}_n) + D(\underset{\sim}{r}_1, \dots, \underset{\sim}{r}, \dots, \underset{\sim}{r}_n)$

3) $D(\underset{\sim}{r}_1, \dots, \underset{\sim}{r}_k, \dots, \underset{\sim}{r}_k, \dots, \underset{\sim}{r}_n) = 0$

4) $D(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n) = 1$

D????