MR2472119 (Review) 05E30 (05C50 05C75)

Lepović, Mirko

Some characterizations of strongly regular graphs. (English summary)

J. Appl. Math. Comput. **29** (2009), *no. 1-2,* 373–381.

A strongly regular graph is a graph of diameter 2 on n vertices that is regular of degree r. Strongly regular graphs are defined by the property that for a pair of vertices distance 1 apart (resp. distance 2 apart) the size of the intersection of their neighbors is constant over all pairs. The size of this intersection is denoted by θ (resp. τ). For a connected strongly regular graph the spectrum is denoted by the three eigenvalues, $\lambda_1 = r$, λ_2 and λ_3, and their three multiplicities, $m_1 = 1$, m_2 and m_3.

The nine Krein parameters of a connected strongly regular graph are usually expressed entirely in terms of the spectrum. This paper gives three alternate versions. The first version gives all nine parameters using just n and the three eigenvalues (i.e., no multiplicities are used). The second version expresses eight of the parameters in terms of the single parameter $\lambda_{22}^{(2)}$ together with the multiplicities (i.e., no eigenvalues). The third version gives all nine parameters using the multiplicities, the difference $\lambda_2 - \lambda_3$ and the difference $\tau - \theta$.

These alternate formulations allow for the classification of several categories of connected strongly regular graphs by their order. Specifically, if n is a prime congruent to $3 \bmod 4$, then there is no strongly regular graph of order n. But if n is a prime congruent to $1 \bmod 4$, then a strongly regular graph of order n is a conference graph (which are characterized by $m_2 = m_3$). If a connected strongly regular graph has order $2k$ and degree r where $2k-1$ is a prime, then the graph is the complement of identical copies of a complete graph on $r+1$ vertices. Finally, a connected strongly regular graph of order $2(2k+1)$, where $2k+1$ is prime, is either a complete bipartite graph, a cocktail-party graph, or one of two other specific strongly regular graphs where k is of the form $l(l+2)$ for an integer l. These characterizations are largely determined by divisibility conditions derived from the alternate expressions for the Krein parameters.

Reviewed by *Robert Beezer*

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MR2455522 (2009g:05098) 05C50 (05C75 05E30)
Bang, Sejeong (KR-PNU); **van Dam, Edwin R.** (NL-TILB-EOR);
Koolen, Jack H. (KR-POST)
Spectral characterization of the Hamming graphs. (English summary)
Linear Algebra Appl. **429** (2008), *no. 11-12,* 2678–2686.

The Hamming graph $H(D,q)$ has vertices that are vectors of length D with entries from an alphabet of size q; two vertices are adjacent if the corresponding vectors differ in exactly one entry. They form a fundamental class of distance-regular graphs of diameter D. This paper is concerned with the characterization of these graphs by their eigenvalues, which is part of the two broader questions of classifying distance-regular graphs and characterizing general graphs by their eigenvalues.

There are many interesting technical results in this paper, but the two main results are as follows. First, suppose q is large, meaning specifically that $2q > D^4 + 2D^3 + 2D^2 - 5D - 4$. Then any graph that is cospectral with $H(D,q)$ is locally the disjoint union of D cliques of size $q-1$. Second, this first result can be employed to show that if $q \geq 36$ then $H(3,q)$ is characterized by its spectrum. There are four graphs cospectral with $H(3,3)$ and two graphs cospectral with $H(3,4)$, but for $5 \leq q < 36$ there are no known cospectral pairs involving $H(3,q)$, so perhaps there is room for improvement in this bound, or an opportunity to discover new cospectral pairs.

This paper is an excellent example of the powerful interplay of algebraic and graph-theoretic techniques and as noted above contains a variety of interesting results which could prove useful for similar investigations. The authors discuss some possible directions (and possible dead ends) for such further investigations.

Reviewed by *Robert Beezer*

References

1. S. Bang, J.H. Koolen, Graphs cospectral with $H(3,q)$ which are locally disjoint union of at most three complete graphs, submitted for publication.
2. N. Biggs, Algebraic Graph Theory, Cambridge Tracts in Mathematics, Cambridge University Press, London, 1974. MR0347649 (50 #151)
3. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989. MR1002568 (90e:05001)
4. E.R. van Dam, W.H. Haemers, J.H. Koolen, E. Spence, Characterizing distance-regularity of graphs by the spectrum, J. Combinatorial Th. A 113 (2006) 1805–1820. MR2269558 (2008f:05210)
5. W.H. Haemers, E. Spence, Graphs cospectral with distance-regular graphs, Linear Multilin. Alg. 39 (1995) 91–107. MR1374473 (96k:05132)
6. A.J. Hoffman, On the polynomial of a graph, Amer. Math. Monthly 70 (1963) 30–36. MR0156333 (27 #6257)
7. W. Mantel, Vraagstuk XXVIII, Wiskundige Opgaven met de Oplossingen 10 (1907) 60–61.
8. K. Metsch, Improvement of Bruck's completion theorem, Designs Codes Crypt. 1 (1991) 99–116. MR1120871 (92m:51014)

9. K. Metsch, A characterization of Grassmann graphs, European J. Combin. 16 (1995) 639–644. MR1356853 (96g:05116)
10. B. Douglas, West, Introduction to Graph Theory, Prentice Hall, Inc., Upper Saddle River, NJ, 1996. MR1367739 (96i:05001)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2443284 (Review) 05C12 (05C50 05E30)
Jurišić, Aleksandar (SV-LJUBC); **Terwilliger, Paul** (1-WI)
Pseudo 1-homogeneous distance-regular graphs. (English summary)
J. Algebraic Combin. **28** (2008), *no. 4,* 509–529.

This paper extends an approach developed in A. Jurišić, J. H. Koolen and P. M. Terwilliger [J. Algebraic Combin. **12** (2000), no. 2, 163–197; MR1801230 (2002c:05164)]. The authors' abstract provides a concise description:

Summary: "Let Γ be a distance-regular graph with diameter $d \geq 2$ and intersection number $a_1 \neq 0$. Let θ be a real number. A pseudo cosine sequence for θ is a sequence of real numbers $\sigma_0, \ldots, \sigma_d$ such that $\sigma_0 = 1$ and $c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i$ for all $i \in \{0, \ldots, d-1\}$. Furthermore, a pseudo primitive idempotent for θ is $E_\theta = s\sum_{i=0}^{d} \sigma_i A_i$, where s is any nonzero scalar and A_i is the ith distance matrix of Γ. Let $\hat{v}$ be the characteristic vector of a vertex $v \in V\Gamma$. For an edge xy of Γ and the characteristic vector w of the set of common neighbours of x and y, we say that the edge xy is tight with respect to θ whenever $\theta \neq k$ (k is the valency of Γ) and a nontrivial linear combination of vectors $E\hat{x}$, $E\hat{y}$ and Ew is contained in $\mathrm{Span}\{\hat{z} | z \in V\Gamma,\ \partial(z,x) = d = \partial(z,y)\}$. When an edge of Γ is tight with respect to two distinct real numbers, a parameterization with $d+1$ parameters of the members of the intersection array of Γ is given (using the pseudo cosines $\sigma_1, \ldots, \sigma_d$, and an auxiliary parameter ε).

"Let S be the set of all the vertices of Γ that are not at distance d from both adjacent vertices x and y. The graph Γ is pseudo 1-homogeneous with respect to xy whenever the distance partition of S corresponding to the distances from x and y is equitable in the subgraph induced on S. We show Γ is pseudo 1-homogeneous with respect to the edge xy if and only if the edge xy is tight with respect to two distinct real numbers. Finally, let us fix a vertex x of Γ. Then the graph Γ is pseudo 1-homogeneous with respect to any edge xy, and the local graph of x is connected if and only if there is the above parameterization with $d+1$ parameters $\sigma_1, \ldots, \sigma_d,\ \varepsilon$ and the local graph of x is strongly regular with nontrivial eigenvalues $a_1\sigma/(1+\sigma)$ and $(\sigma_2 - 1)/(\sigma - \sigma_2)$."

Reviewed by *Robert Beezer*

References

1. Bannai, E., Ito, T.: Algebraic Combinatorics I: Association Schemes. Benjamin-Cummings, California (1984) MR0882540 (87m:05001)
2. Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin, Heidelberg (1989) MR1002568 (90e:05001)
3. Curtin, B., Nomura, K.: 1-homogeneous, pseudo-1-homogeneous, and 1-thin distance-regular graphs. J. Comb. Theory Ser. B **93**, 279–302 (2005) MR2117939 (2006c:05145)
4. Go, J.T., Terwilliger, P.M.: Tight distance-regular graphs and the subconstituent algebra. Eur. J. Comb. **23**, 793–816 (2002) MR1932680 (2003i:05138)
5. Godsil, C.D.: Algebraic Combinatorics. Chapman and Hall, New York (1993) MR1220704 (94e:05002)
6. Jurišić, A., Koolen, J.: A local approach to 1-homogeneous graphs. Des. Codes Cryptogr. **21**, 127–147 (2000) MR1801195 (2001m:05265)
7. Jurišić, A., Koolen, J., Terwilliger, P.: Tight Distance-Regular Graphs. J. Algebr. Comb. **12**, 163–197 (2000) MR1801230 (2002c:05164)
8. Pascasio, A.A.: Tight graphs and their primitive idempotents. J. Algebr. Comb. **10**, 47–59 (1999) MR1701283 (2000g:05155)
9. Terwilliger, P.M.: The subconstituent algebra of a distance-regular graph; thin modules with endpoint one. Linear Algebra Appl. **356**, 157–187 (2002) MR1944685 (2003j:05135)
10. Terwilliger, P.M.: An inequality involving the local eigenvalues of a distance-regular graph. J. Algebr. Comb. **19**, 143–172 (2004) MR2058282 (2005d:05156)
11. Terwilliger, P.M., Weng, C.: Distance-regular graphs, pseudo primitive idempotents, and the Terwilliger algebra. Eur. J. Comb. **25**(2), 287–298 (2004) MR2070549 (2005c:05202)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2431756 (2009d:05263) 05E30 (05C75)

Pan, Yeh-jong (RC-NCT-DA); **Weng, Chih-wen** (RC-NCT-DA)

3-bounded property in a triangle-free distance-regular graph. (English summary)

European J. Combin. **29** (2008), *no. 7,* 1634–1642.

Many properties of a distance-regular graph of diameter d can be described by a certain set of $2d-1$ of its intersection numbers. For certain classes of graphs, these numbers can be described with just 4 parameters. These graphs are then said to have “classical parameters”. This article is concerned with distance-regular graphs having classical parameters where the intersection numbers a_1 and a_2 satisfy $a_1 = 0$ and $a_2 \neq 0$.

Let $\partial(x,y)$ denote the distance between vertices u and v. Then the sequence x, y, z is weak-geodetic if $\partial(x,y)+\partial(y,z) \leq \partial(x,z)+1$. A subset of vertices W is weak-geodetically closed if given any weak-geodetic sequence x, y, z of the graph $x, z \in W$, it follows that $y \in W$. Finally, a graph is i-bounded if given vertices x, y with $\partial(x,y) \leq i$, there is a regular subgraph of diameter $\partial(x,y)$ containing x and y that is weak-geodetically closed. In [European J. Combin. **18** (1997), no. 2, 211–229; MR1429246 (97m:05265)] the second author used this property to further restrict the values of the classical parameters for a wide class of distance-regular graphs.

The main result of this paper is that the distance-regular graphs described in the first paragraph are 3-bounded. The proof is constructive, in the sense that the required regular weak-geodetically closed subgraph of diameter 3 is described explicitly.

The authors conclude by remarking that the 4-bounded property seems much harder to prove, but they believe the 3-bounded property to be sufficient for classifying all of the distance-regular graphs with classical parameters, $a_1 = 0$ and $a_2 \neq 0$.

Reviewed by *Robert Beezer*

References

1. E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, 1984. MR0882540 (87m:05001)
2. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989. MR1002568 (90e:05001)
3. A.E. Brouwer, H.A. Wilbrink, The structure of near polygons with quads, Geometriae Dedicata 14 (1883) 145–176. MR0708631 (85b:05045)
4. A. Hiraki, Strongly closed subgraphs in a regular thick near polygon, European Journal of Combinatorics 20 (8) (1999) 789–796. MR1730825 (2001f:05100)
5. A.A. Ivanov, S.V. Shpectorov, Characterization of the association schemes of Hermitian forms over $GF(2^2)$, Geometriae Dedicata 30 (1989) 23–33. MR0995936 (90d:05061)
6. Y. Pan, M. Lu, C. Weng, Triangle-free distance-regular graphs, Journal of Algebraic Combinatorics (in press). http://www.springerlink.com/content/11180j3068130238/.
7. E.E. Shult, A. Yanushka, Near n-gons and line systems, Geometriae Dedicata 9 (1980) 1–72. MR0566437 (82b:51018)
8. H. Suzuki, On strongly closed subgraphs of highly regular graphs, European Journal of Combinatorics 16 (1995) 197–220. MR1324430 (96m:05200)
9. H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five, Kyushu Journal of Mathematics 50 (2) (1996) 371–384. MR1447927 (98g:05160)
10. P. Terwilliger, The subconstituent algebra of an association scheme (part I), Journal of Algebraic Combinatorics 1 (1992) 363–388. MR1203683 (94b:05222)
11. P. Terwilliger, A new inequality for distance-regular graphs, Discrete Mathematics 137 (1995) 319–332. MR1312463 (95k:05187)
12. C. Weng, Weak-geodetically closed subgraphs in distance-regular graphs, Graphs and Combinatorics 14 (1998) 275–304. MR1645986 (99f:05129)
13. C. Weng, D-bounded distance-regular graphs, European Journal of Combinatorics 18 (1997) 211–229. MR1429246 (97m:05265)

14. C. Weng, Classical distance-regular graphs of negative type, Journal of Combinatorial Theory, Series B 76 (1999) 93–116. MR1687322 (2000d:05130)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2413879 (2009d:05261) 05E30 (11T30 15A63)
Feng, Rongquan (PRC-BJ-MAM); **Wang, Yangxian** (PRC-HNU-MI);
Ma, Changli (PRC-HNU-MI); **Ma, Jianmin** (1-COS)
Eigenvalues of association schemes of quadratic forms. (English summary)
Discrete Math. **308** (2008), *no. 14,* 3023–3047.

Quadratic forms and symmetric bilinear forms over finite fields can be used to construct association schemes, as first described in [Y. Egawa, J. Combin. Theory Ser. A **38** (1985), no. 1, 1–14; MR0773550 (86f:05018)]. In this article, the authors continue their work investigating these structures for a greater range of parameters.

Specifically, recursive formulas are given for the eigenvalues of $\mathrm{Qua}(n,q)$, the association scheme built from quadratic forms on n variables over the finite field F_q with even q. These are used to give the eigenvalues (as functions of q) for $\mathrm{Qua}(n,q)$, and two related fusion schemes, when $n=2,3,4$.

Reviewed by *Robert Beezer*

References

1. E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjaming, CA, 1984. MR0882540 (87m:05001)
2. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Berlin, 1989. MR1002568 (90e:05001)
3. Y. Egawa, Association schemes of quadratic forms, J. Combin. Theory A 38 (1985) 1–14. MR0773550 (86f:05018)
4. C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, London, 1993. MR1220704 (94e:05002)
5. A. Munemasa, The Geometry of Orthogonal Groups over Finite Fields, Lecture Note in Mathematics, vol. 3, Sophia University, Tokyo, Japan, 1996.
6. A. Munemasa, D.V. Pasechnik, S.V. Shpectorov, The automorphism group and the convex subgraphs of the quadratic forms graph in characteristic 2, J. Algebraic Combin. 2 (1993) 411–419. MR1241509 (94j:05138)
7. Z. Wan, Geometry of Classical Groups over Finite Fields, Studentlitteratur, Lund, 1993.

MR1254440 (95a:51027)
8. Y. Wang, J. Ma, Association schemes of symmetric matrices over a finite field of characteristic two, J. Statist. Plann. Inference 51 (1996) 351–371. MR1397542 (97d:05285)
9. Y. Wang, C. Wang, C. Ma, Association schemes of quadratic forms over a finite field of characteristic two, Chinese Sci. Bull. 43 (1998) 1965–1968. MR1667250 (99j:05200)
10. Y. Wang, C. Wang, C. Ma, J. Ma, Association schemes of quadratic forms and symmetric bilinear forms, J. Algebraic Combin. 17 (2003) 149–161. MR1971743 (2004c:15047)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2338247 (2008h:05102) 05C75 (05E30)
Paduchikh, D. V. (RS-AOSUR-A)
The nonexistence of locally $\overline{J}(10,5)$-graphs. (English summary)
Proc. Steklov Inst. Math. **257** (2007), *suppl. 1,* *S*155–*S*163.

The graph $\overline{J}(10,5)$ is a standard quotient of a Johnson graph. To construct the graph, begin with the Johnson graph, $J(10,5)$, whose vertices are all the 5-element subsets of a 10-set, with two vertices adjacent if and only if their 5-sets intersect in a 4-set. The quotient is the derived graph on half as many vertices where two vertices are identified if their corresponding sets are complementary. $\overline{J}(10,5)$ is known to be strongly regular with parameters $(126,25,8,4)$.

Given a graph H, the graph G is said to be a "locally H-graph" if for every vertex v of G, the set of vertices adjacent to v induces a subgraph isomorphic to H. The purpose of this paper is to prove that there is no graph that is a locally $\overline{J}(10,5)$-graph.

This result is a part of a broader investigation by the author and his colleagues. Given two vertices a distance two apart, form the subgraph induced by the intersection of the immediate neighbors of each vertex. This study asks which graphs result from restrictions on the structure of these subgraphs.

Reviewed by *Robert Beezer*

References

1. A. A. Makhnev, Proc. Steklov Inst. Math., Suppl. 2 (2001), pp. 169–178. MR2067928 (2005a:05190)
2. V. V. Kabanov, A. A. Makhnev, and D. V. Paduchikh, in *Proceedings of Intern. Conf. "Algebra and Its Application"* (Izd. Krasnoyarsk. Univ., 2002), pp. 55–56.
3. A. Blokhuis and A. E. Brouwer, J. Graph Theory **13** (2), 229 (1989). MR0994744 (90h:05096)

MR2337546 (2008g:05226) 05E30 (05C50 05C75)

Khodkar, Abdollah (1-WGA); **Leach, David [Leach, C. D.]** (1-WGA);
Robinson, David [Robinson, David Guy] (1-WGA)

Every $(2, r)$-regular graph is regular. (English summary)

Util. Math. **73** (2007), 169–172.

A graph is (t, r)-regular if for every set S of t vertices, the cardinality of the union of the neighbors of the vertices in S equals r [T. W. Haynes and L. R. Markus, Util. Math. **59** (2001), 155–165; MR1832609 (2001m:05141)]. As the title indicates, this paper classifies the $(2, r)$-regular graphs. So in these graphs, every pair of distinct vertices has a total of r neighbors, where we count common neighbors just once.

The main result of this paper is that a $(2, r)$-regular graph is a member of the well-known class of strongly regular graphs. In particular the graph will be regular, and if n is the number of vertices, then the degree is $\frac{1}{2}(2n - 1 - \sqrt{4(n-1)(n-r)+1})$. So as a corollary, if there exists a $(2, r)$-regular graph on n vertices, then the quantity $4(n-1)(n-r)+1$ must be a perfect square.

Techniques employed include the construction of a pairwise balanced design derived from a $(2, r)$-regular graph and a theorem of H. J. Ryser about $(0, 1)$ matrices with constant row and column sums.

Reviewed by *Robert Beezer*

MR2186704 (2006j:05091) 05C25 (20B25)
Feng, Yan-Quan (PRC-BJTU); **Xu, Ming-Yao** (PRC-BJ)
Automorphism groups of tetravalent Cayley graphs on regular p-groups. (English summary)
Discrete Math. **305** (2005), *no. 1-3,* 354–360.

A Cayley graph Γ is constructed with a group G as its vertex set, and directed edges are built from the elements of a subset S of G. The resulting graph is always vertex-transitive, since an element of G can multiply each vertex on the right to create an automorphism of the graph. The collection of all such automorphisms is the right regular representation, $R(\Gamma)$, a subgroup of the full automorphism group $\mathrm{Aut}(\Gamma)$. Another subgroup of automorphisms of Γ is $\mathrm{Aut}(G,S)=\{\alpha\in \mathrm{Aut}(G)\mid S^\alpha=S\}$. The Cayley graph Γ is said to be normal on G if the right regular representation $R(G)$ is a normal subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$.

The normality of Cayley graphs has been consider by many, including the authors of this paper and others such as Alspach, Fang, Klin, Pöschel, Li, Praeger, and Wang. As an indicator of the difficulty of this problem, consider that two Cayley graphs can be isomorphic while being constructed from non-isomorphic groups, and hence one can be normal, and the other not. The paper under review contains a short, comprehensive survey of known results in this area.

This paper shows that if $p\neq 2,5$, then all connected tetravalent Cayley graphs on regular p-groups are normal. This is accomplished by employing the classification of the finite simple groups to show that the automorphism group of Γ is a semi-direct product, $\mathrm{Aut}(\Gamma)=R(G)\rtimes \mathrm{Aut}(G,S)$. This paper is a pleasant mix of detailed and intertwined arguments from both graph theory and group theory.

Reviewed by *Robert Beezer*

References

1. B. Alspach, Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree, J. Combin. Theory 15 (1973) 12–17. MR0332553 (48 #10880)
2. Y.G. Baik, Y.-Q. Feng, H.S. Sim, The normality of Cayley graphs of finite Abelian groups with valency 5, Systems Sci. Math. Sci. 13 (2000) 425–431. MR1790498 (2001e:05052)
3. Y.G. Baik, Y.-Q. Feng, H.S. Sim, M.Y. Xu, On the normality of Cayley graphs of Abelian groups, Algebra Colloq. 5 (1998) 297–304. MR1679566 (2000a:05102)
4. N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993. MR1271140 (95h:05105)
5. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Oxford University Press, Oxford, 1985 (Currently available on the world-wide web at the URL

   $$<$$

   http://web.mat.bham.ac.uk/atlas/v2.0/

   $$>$$

) MR0827219 (88g:20025)
6. E. Dobson, D. Witte, Transitive permutation groups of prime-squared degree, J. Algebraic

Combin. 16 (2002) 43–69. MR1941984 (2004c:20007)
7. S.F. Du, R.J. Wang, M.Y. Xu, On the normality of Cayley digraphs of order twice a prime, Australasian J. Combin. 18 (1998) 227–234. MR1658278 (99g:05095)
8. X.G. Fang, C.H. Li, J. Wang, M.Y. Xu, On cubic Cayley graphs of finite simple groups, Discrete Math. 244 (2002) 67–75. MR1881684 (2002k:05107)
9. Y.-Q. Feng, J.H. Kwak, R.J. Wang, Automorphism groups of 4-valent connected Cayley graphs of p-groups, Chinese Ann. Math. 22B (2001) 281–286. MR1845749 (2002f:05083)
10. Y.-Q. Feng, D.J. Wang, J.L. Chen, A family of nonnormal Cayley digraphs, Acta Math. Sinica (N.S.) 17 (2001) 147–152. MR1831753 (2002d:05062)
11. Y.-Q. Feng, R.J. Wang, M.Y. Xu, Automorphism groups of 2-valent connected Cayley digraphs on regular p-groups, Graphs Combin. 18 (2002) 253–257. MR1913667 (2003e:20004)
12. C.D. Godsil, On the full automorphism group of a graph, Combinatorica 1 (1981) 243–256. MR0637829 (83a:05066)
13. D. Gorenstein, Finite Simple Groups, Plenum Press, New York, 1982. MR0698782 (84j:20002)
14. B. Huppert, Endliche gruppen I, Springer, Berlin, 1979. MR0224703 (37 #302)
15. R.H. Klin, R. Pöschel, The König problem, the isomorphism problem for circulant graphs and the method of Schur, Proceedings of the International Colloquium on Algebraic Methods in Graph Theory, Szeged, 1978, Coll. Mat. Soc. János Bolyai 27.
16. M.Ch. Klin, R. Pöschel, The isomorphism problem for circulant graphs with p^n vertices, Preprint P-34/80 ZIMM, Berlin, 1980.
17. C.H. Li, On isomorphisms of connected Cayley graphs III, Bull. Austral. Math. Soc. 58 (1998) 137–145. MR1633776 (99g:05100)
18. A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998) 203–218. MR1603687 (98k:57006)
19. C.E. Praeger, Finite normal edge-transitive graphs, Bull. Austral. Math. Soc. 60 (1999) 207–220. MR1711938 (2000j:05057)
20. D.J. Robinson, A Course in the Theory of Groups, Springer, New York, 1982. MR0648604 (84k:20001)
21. G. Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804. MR0097068 (20 #3548)
22. W.R. Scott, Group Theory, Dover Press, New York, 1987. MR0896269 (88d:20001)
23. M. Suzuki, Group Theory I, Springer, New York, 1982. MR0648772 (82k:20001c)
24. C.Q. Wang, D.J. Wang, M.Y. Xu, On normal Cayley graphs of finite groups, Sci. China(A) 28 (1998) 131–139.
25. H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964. MR0183775 (32 #1252)
26. M.Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, Discrete math. 182 (1998) 309–319. MR1603719 (98i:05096)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2186692 (2006j:05218) 05E30 (05C25)
Leifman, Yefim I. (IL-BILN-CS); **Muzychuk, Mikhail E.**
Strongly regular Cayley graphs over the group $\mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^n}$. (English summary)
Discrete Math. **305** (2005), *no. 1-3,* 219–239.

A strongly regular graph is a distance-regular graph of diameter 2. More precisely, a (v, k, λ, μ)-strongly regular graph is a regular graph of degree k on v vertices such that (1) each adjacent pair of vertices has exactly λ common neighbors, and (2) each pair of non-adjacent vertices has exactly μ common neighbors. A Cayley graph arises from a construction that obtains vertices from elements of a group, while edges are derived from elements of a generating set for the group. The resulting graph is always vertex-transitive.

This paper classifies all of the strongly regular Cayley graphs formed from the group $Z_{p_n} \oplus Z_{p_n}$, where p is an odd prime. A principal tool used in this paper is the theory of S-rings. In particular, the existence of a strongly regular Cayley graph formed from a group G then implies the existence of an S-ring over G. Various theorems about the existence of S-rings then bolster the argument that the first "interesting" family of groups for which to consider the possibility of strongly regular Cayley graphs is indeed $Z_{p_n} \oplus Z_{p_n}$. The nontrivial graphs resulting from the classification are of Latin square type, with eigenvalues k, $\frac{k}{1-p_n}$ and $p^n + \frac{k}{1-p_n}$. A corollary yields a complete classification of the strongly regular Cayley graphs with Paley parameters over an abelian group of rank 2.

These results extend previous results of K. H. Leung and S. L. Ma [Bull. London Math. Soc. **27** (1995), no. 6, 553–564; MR1348709 (96g:05025); S. L. Ma, Ars Combin. **27** (1989), 211–220; MR0989440 (90a:05035)] while relying heavily on techniques and results of W. G. Bridges and R. A. Mena [Ars Combin. **8** (1979), 143–161; MR0557072 (81b:05022); J. Combin. Theory Ser. A **32** (1982), no. 2, 264–280; MR0654627 (83g:05037)].

Reviewed by *Robert Beezer*

References

1. R.C. Bose, Strongly regular graphs, partial geometries, and partially balanced designs, Pacific J. Math. 13 (1963) 389–419. MR0157909 (28 #1137)
2. W.G. Bridges, R.A. Mena, Rational circulants with rational spectr and cyclic strongly regular graphs, Ars Combin. 8 (1979) 143–161. MR0557072 (81b:05022)
3. W.G. Bridges, R.A. Mena, Rational G-matrices with rational eigenvalues, J. Combin. Theory Ser. A 32 (1982) 264–280. MR0654627 (83g:05037)
4. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-regular graphs, Springer, Berlin, 1989. MR1002568 (90e:05001)
5. J.A. Davis, Partial difference sets in p-groups, Arch. Math. 63 (1994) 103–110. MR1289290 (96d:05021)

6. P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Reports (Suppl. 10) (1973) 1–97. MR0384310 (52 #5187)
7. I.A. Faradzev, A.A. Ivanov, M.H. Klin, Galois correspondence between permutation groups and cellular rings (association schemes), Graphs Combin. 6 (1992) 202–224. MR1092582 (92d:05182)
8. J.J. Golfand, A.A. Ivanov, M.H. Klin, Amorphic cellular rings, in: I.A. Faradzev, A.A. Ivanov, M.H. Klin, A.J. Woldar (Eds.), Investigations in Algebraic Theory of Combinatorial Objects, Mathematics and Its Applications (Soviet Series), vol. 84, Kluwer Academic Publishers, Dordrecht, 1994, pp. 167–187. MR1273365 (94m:05004)
9. W.H. Haemers, E. Spence, The pseudo-geometric graphs for generalized quadrangle of order $(3, t)$, European J. Combin. 22 (6) (2001) 839–845. MR1848328 (2002i:51004)
10. R. Kochendorfer, Untersuchungen über eine Vermutung von W. Burnside, Schr. Math. Sem. Inst. Angew. Math. Univ. Berlin 3 (1937) 155–180.
11. K.H. Leung, S.L. Ma, Partial difference sets with Paley parameters, Bull. London Math. Soc. 27 (1995) 553–564. MR1348709 (96g:05025)
12. S.L. Ma, On association schemes, Schur rings, strongly regular graphs and partial difference sets, Ars Combin. 27 (1989) 211–220. MR0989440 (90a:05035)
13. J.J. Seidel, Strongly regular graphs, in: W.T. Tutte (Ed.), Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 185–197. MR0253935 (40 #7148)
14. H. Wielandt, Finite Permutation Groups, Academic Press, New York, 1964. MR0183775 (32 #1252)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2167477 (2006e:05137) 05C70 (05C05 05C50)
Ciucu, Mihai (1-GAIT); **Yan, Weigen** (PRC-JMU-SS); **Zhang, Fuji** (PRC-XIAM)
The number of spanning trees of plane graphs with reflective symmetry. (English summary)
J. Combin. Theory Ser. A **112** (2005), *no. 1,* 105–116.

Suppose a planar graph is drawn on the plane without crossings and there exists a line such that the embedding of the graph is invariant after a reflection about the line. This article studies such graphs when every edge of the graph is either disjoint from the line of symmetry, or lies entirely on the line of symmetry. In this case, the authors show that the number of spanning trees of the symmetric graph may be expressed as a product of the number of spanning trees of two different graphs derived from the two (isomorphic) "halves" of the original graph, along with an additional factor that is a power of 2.

Two proofs are given. The first is algebraic and involves adjacency matrices and dual graphs, while the second is combinatorial. Because of a well-known relationship between the number of spanning trees in a graph G and the number of perfect matchings of a certain graph derived from G, the combinatorial proof builds on the first author's earlier results on perfect matchings in graphs with reflective symmetry [M. Ciucu, J. Combin. Theory Ser. A **77** (1997), no. 1, 67–97; MR1426739 (98a:05112)]. (The reader should be aware that several citations in the paper lead to the wrong entry in the references.)

Reviewed by *Robert Beezer*

References

1. N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993. MR1271140 (95h:05105)
2. J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, American Elsevier, New York, 1976. MR0411988 (54 #117)
3. T. Chow, The Q-spectrum and spanning trees of tensor products of bipartite graphs, Proc. Amer. Math. Soc. 125 (1997) 3155–3161. MR1415578 (97m:05177)
4. M. Ciucu, Perfect matchings, spanning trees, plane partitions and statistical physics, Ph.D. Thesis, University of Michigan, Ann Arbor, MI, 1996.
5. M. Ciucu, Enumeration of perfect matchings in graphs with reflective symmetry. J. Combin. Theory Ser. A 77 (1997) 67–97. MR1426739 (98a:05112)
6. D. Cvetković, M. Doob, I. Gutman, Torĝasev, Recent results in the theory of graph spectra, Annals of Discrete Mathematics, vol. 36, North-Holland, Amsterdam, 1988 (Theorem 3.34). MR0926481 (89d:05130)
7. D. Cvetković, I. Gutman, A new method for determining the number of spanning trees, Publ Inst. Math. (Beograd) 29 (1981) 49–52. MR0657093 (83f:05046)
8. S.J. Cyvin, I. Gutman, Kekule structures in Benzennoid Hydrocarbons, Springer, Berlin, 1988.
9. D.E. Knuth, Aztec diamonds, checkerboard graphs, and spanning trees, J. Algebra Combinatories 6 (1997) 253–257. MR1456581 (98h:05132)
10. R. Lyons, Asymptotic enumeration of spanning trees, arXiv:math.CO/0212165 v1 11 Dec 2002. MR2160416 (2006j:05048)
11. L. Lovász, Combinatorial Problems and Exercises, second ed., Budapest, Akadémiai Kiadó, 1993. MR1265492 (94m:05001)
12. R. Shrock, F.Y. Wu, Spanning trees on graphs and lattices in d dimensions, J. Phys. A 33 (2000) 3881–3902. MR1769549 (2001b:05111)
13. R. Stanley, Spanning trees of Aztec diamond, Discrete Math. 157 (1996) 375–388 (Problem 251).
14. R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999. MR1676282 (2000k:05026)
15. H.N.V. Temperley, in: Combinatorics: Being the Proceedings of the Conference on Combinatorial Mathematics held at the Mathematical Institute, Oxford, 1972, pp. 356–357.

MR2128338 (2006a:05180) 05E30 (33C45 33D45)
Terwilliger, Paul (1-WI)

Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array. (English summary)

Des. Codes Cryptogr. **34** (2005), *no. 2-3,* 307–332.

This technical article continues the author's study of Leonard pairs and Leonard systems. A Leonard system is a sequence of matrices chosen from the algebra generated by a single matrix such that the sequence satisfies a list of conditions, which mainly involve eigenvalues and products that must result in tridiagonal matrices. These sequences are motivated by D. A. Leonard's original paper on pairs of matrices [SIAM J. Math. Anal. **13** (1982), no. 4, 656–663; MR0661597 (83m:42014)], distance-regular graphs, association schemes and orthogonal polynomials. An introduction to the topic can be found in [P. M. Terwilliger, J. Comput. Appl. Math. **153** (2003), no. 1-2, 463–475; MR1985715 (2004h:05137)].

The set of isomorphism classes of Leonard systems corresponds (via a bijection) to the set of all parameter arrays, where a parameter array is a sequence of scalars satisfying five arithmetic conditions. This article gives two characterizations of Leonard systems by employing this bijection. One characterization relates to bidiagonal matrices, while the other relates to polynomials. Then, in Section 5, every possible parameter array is listed in the course of presenting thirteen families of parameter arrays. That every possible array is described is the substance of Theorem 5.16. These families are organized by reference to classes of corresponding polynomials.

Reviewed by *Robert Beezer*

References

1. R. Askey and J. A. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols, *SIAM J. Math. Anal.*, Vol. 10 (1979) pp. 1008–1016. MR0541097 (80k:33012)
2. E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London (1984). MR0882540 (87m:05001)
3. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, Vol. 35. Cambridge University Press, Cambridge (1990). MR1052153 (91d:33034)
4. Ya. Granovskii, I. Lutzenko and A. Zhedanov, Mutual integrability, quadratic algebras, and

dynamical symmetry, *Ann. Physics*, Vol. 217, No. 1 (1992) pp. 1–20. MR1173277 (93k:81072)
5. F. A. Grunbaum and L. Haine, A q-version of a theorem of Bochner, *J. Comput. Appl. Math.*, Vol. 68, No. 1–2 (1996) pp. 103–114. MR1418753 (97m:33005)
6. T. Ito, K. Tanabe and P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, In *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI (2000). MR1816397 (2002h:05162)
7. R. Koekoek and R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polyomials and its q-analog*, Vol. 98–17 of *Reports of the faculty of Technical Mathematics and Informatics*, Delft, The Netherlands (1998).
8. H. T. Koelink, Askey-Wilson polynomials and the quantum $\mathrm{su}(2)$ group: survey and applications, *Acta Appl. Math.*, Vol. 44, No. 3 (1996) pp. 295–352. MR1407326 (98k:33037)
9. D. Leonard, Orthogonal polynomials, duality, and association schemes, *SIAM J. Math. Anal.*, Vol. 13, No. 4 (1982), pp. 656–663. MR0661597 (83m:42014)
10. H. Rosengren, *Multivariable Orthogonal Polynomials as Coupling Coefficients for Lie and Quantum Algebra Representations*, Centre for Mathematical Sciences, Lund University, Sweden (1999).
11. J. J. Rotman, *Advanced Modern Algebra*, Prentice-Hall, Saddle River NJ (2002). MR2043445 (2005b:00002)
12. P. Terwilliger, The subconstituent algebra of an association scheme, *J. Algebraic Combin.* Vol. 1, No. 4 (1992) pp. 363–388. MR1203683 (94b:05222)
13. P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, *Linear Algebra Appl.*, Vol. 330 (2001) pp. 149–203. MR1826654 (2002h:15021)
14. P. Terwilliger, Two relations that generalize the q-Serre relations and the Dolan-Grady relations, In *Proc. of Nagoya 1999 Workshop on Physics and Combinatorics (Nagoya, Japan 1999)*, World Scientific Publishing Co., Inc., River Edge, NJ, Providence RI (2000). MR1865045 (2003a:16044)
15. P. Terwilliger, Leonard pairs from 24 points of view, *Rocky Mountain J. Math.*, Vol. 32, No. 2 (2002), pp. 1–62. MR1934918 (2003m:05218)
16. P. Terwilliger, Introduction to Leonard pairs. *OPSFA Rome 2001, J. Comput. Appl. Math.* Vol. 153, No. 2 (2003) pp. 463–475. MR1985715 (2004h:05137)
17. P. Terwilliger, Introduction to Leonard pairs and Leonard systems. *Sūrikaisekikenkyūsho Kōkyūroku*, (1109) pp. 67–79, (1999). Algebraic combinatorics (Kyoto, 1999). MR1754413
18. P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other: the TD-D and the LB-UB canonical form. Preprint. cf. MR 2002h:15021
19. P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition. Preprint. cf. MR 2006a:05181
20. A. S. Zhedanov, "Hidden symmetry" of Askey-Wilson polynomials, *Teoret. Mat. Fiz.*, Vol. 89, No. 2 (1991) pp. 190–204. MR1151381 (93f:33019)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2128329 (2006d:05188) 05E30 (05C50)
Bang, Sejeong (J-KYUSGM); **Koolen, J. H.** (KR-POST-CM)
Some interlacing results for the eigenvalues of distance-regular graphs. (English summary)
Des. Codes Cryptogr. **34** (2005), *no. 2-3,* 173–186.

Given an integer $k > 3$, E. Bannai and T. Ito conjectured that there are only finitely many distance-regular graphs with valency k [in *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, 343–349, Proc. Sympos. Pure Math., 47, Part 2, Amer. Math. Soc., Providence, RI, 1987; MR0933424]. For a distance-regular graph with intersection numbers a_i, b_i, c_i, $0 \le i \le d$, define $h(G) = |\{i \mid (c_i, a_i, b_i) = (1, a_1, b_1)\}|$. Then this conjecture is equivalent to the statement that $h(G)$ is bounded above by a function of the valency of G. This statement would appear to be true, since there is no known distance-regular graph (of valency 3 or more) for which $h(G)$ has a value greater than 5.

The end of this paper contains an improvement of an earlier result of J. H. Koolen and V. L. Moulton [J. Algebraic Combin. **19** (2004), no. 2, 205–217; MR2058285 (2005c:05198)]. This new result says that if G is a triangle-free distance-regular graph with large valency and a "large" nontrivial eigenvalue, then $h(G)$ can be bounded above by a constant, independent of the valency. This is the motivation for the main results of this paper, which give rather restrictive consequences when a distance-regular graph has a large nontrivial eigenvalue.

Suppose a distance-regular graph of valency k has all but its largest eigenvalue bounded in absolute value by a number κ, with $0 < \kappa \le k/3$. This is the notion of a "large" nontrivial eigenvalue mentioned above. In this case, the authors prove a restrictive condition on the intersection numbers a_i, namely $\varnothing \ne \{i \mid a_i > \frac{k-\kappa}{2}\} \subseteq \{i \mid a_i > \kappa\} \subseteq \{j, j+1\}$ for some j. For the case of a bipartite distance-regular graph, entirely similar hypotheses yield an identical conclusion about the differences $c_{i+1} - c_i$ (rather than the values of a_i).

Reviewed by *Robert Beezer*

References

1. E. Bannai and T. Ito, The study of distance-regular graphs from the algebraic (i.e. character theoretical) viewpoint, *Proceedings of Symposia in Pure Mathematics*, Vol. 47 (1987) pp. 343–349. MR0933424
2. Norman Biggs, *Algebraic Graph Theory*, Second edition, Cambridge University Press, Cambridge, (1993). MR1271140 (95h:05105)
3. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, (1989). MR1002568 (90e:05001)
4. C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall Mathematics Series, Chapman and Hall, New York, (1993). MR1220704 (94e:05002)
5. Willem H. Haemers, Interlacing Eigenvalues and Graphs, *Linear Algebra and Application*, Vol.

226/228 (1995) pp. 593–616. MR1344588 (96e:05110)

6. A. A. Ivanov, Bounding the diameter of a distance-regular graph., (Russian) *Dokl. Akad. Nauk SSSR*, Vol. 271, No. 4 (1983) pp. 789–792. MR0719819 (84m:05045)
7. J. H. Koolen and V. Moulton, There are finitely many triangle-free distance-regular graphs with degree 8, 9, or 10. J. Algebraic Combin. 19 (2004). No. 2, 205–217. MR2058285 (2005c:05198)
8. P. Terwilliger, Eigenvalue multiplicities of highly symmetric graphs, *Discrete Mathmatics*, Vol. 41 (1982) pp. 295–302. MR0676891 (83m:05100)
9. P. Terwilliger, Distance-regular graphs and (s, a, c, k)-graphs, *J. Combinatorial Theory (B)*, Vol. 34 (1983) pp. 151–164. MR0703600 (84i:05100)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2112881 (2005k:05134) 05C50 (05C70 05E30)

Brouwer, Andries E. (NL-EIND); **Haemers, Willem H.** (NL-TILB-EOR)

Eigenvalues and perfect matchings. (English summary)

Linear Algebra Appl. **395** (2005), 155–162.

This article describes sufficient conditions, in terms of the eigenvalues of the adjacency and Laplacian matrices, for a graph to have a perfect matching. These results build upon, and expand, results due to Frobenius, König, Hall, and Tutte. For example, suppose that a bipartite graph has n vertices and the eigenvalues of the adjacency matrix are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then a result of G. Frobenius [Sitzungsber. K. Preuss. Akad. Wissen. Berlin (1917), 274–277; JFM 46.0144.05] can be used to formulate a result that says that if $n\lambda_1 = \sum_{i=1}^n \lambda_i^2$, then the graph has a perfect matching.

Results progress from arbitrary graphs, through regular graphs and then distance-regular graphs. First, suppose that a graph has an even number of vertices, n, and the Laplacian matrix has eigenvalues $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. If $\mu_n \leq 2\mu_2$, then the graph has a perfect matching.

Suppose that a graph is regular of degree r, n is even, and $\lambda_3 \leq r - 1 + \frac{3}{r+1}$ (for r even) or $\lambda_3 \leq r - 1 + \frac{3}{r+2}$ (for r odd), then the graph has a perfect matching. In terms of the Laplacian matrix, this result provides the following corollary. Suppose that a graph is regular with an even number of vertices and $\mu_2 \geq 1$, then the graph has a perfect matching.

The authors comment that they do not know of any distance-regular graph that does not satisfy the condition for regular graphs. However, rather than taking this approach, they prove that every distance-regular graph of degree r is r-edge-connected. It then follows by results in [G. Chartrand, D. L. Goldsmith and S. Schuster, Colloq. Math. **41** (1979), no. 2, 339–344; MR0591941 (82a:05070)] that every distance-regular graph on an even number of vertices has a perfect matching.

Reviewed by *Robert Beezer*

References

1. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Heidelberg, 1989. MR1002568 (90e:05001)
2. A.E. Brouwer, D.M. Mesner, The connectivity of strongly regular graphs, European J. Combin. 6 (1985) 215–216. MR0818594 (87i:05132)
3. R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge Univ. Press, 1991. MR1130611 (93a:05087)
4. G. Chartrand, D.L. Goldsmith, S. Schuster, A sufficient condition for graphs with 1-factors, Colloq. Math. 41 (1979) 339–344. MR0591941 (82a:05070)
5. D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, third ed., Johann Abrosius Barth Verlag, 1995 (First edition: Deutscher Verlag der Wissenschaften, Berlin 1980; Academic Press, New York 1980). MR1324340 (96b:05108)
6. G. Frobenius, Über zerlegbare Determinanten, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1917) 456–477.
7. W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 226–228 (1995) 593–616. MR1344588 (96e:05110)
8. P. Hall, On representations of subsets, J. London Math. Soc. 10 (1935) 26–30.
9. D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 453–465. MR1511872
10. D. König, Graphok és Matrixok (Graphs and matrices), Matematikai és Fizikai Lapok 38 (1931) 116–119.
11. W.T. Tutte, The factorizations of linear graphs, J. London Math. Soc. 22 (1947) 107–111. MR0023048 (9,297d)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2110978 (2005i:05203) 05E30 (05C50)
Terwilliger, Paul (1-WI); **Weng, Chih-wen** (RC-NCT-AM)
An inequality for regular near polygons. (English summary)
European J. Combin. **26** (2005), *no. 2,* 227–235.

Near polygons are a class of distance-regular graphs. Suppose a near polygon is regular of degree k with diameter $d \geq 3$ and has intersection numbers $a_1 > 0$ and $c_2 > 1$. Let θ_1 be the second-largest eigenvalue of the graph; this article shows that $\theta_1 \leq \frac{k-a_1-c_2}{c_2-1}$. Furthermore, the case of equality is characterized by the following three equivalent statements: (i) the bound on θ_1 is an equality, (ii) the graph is Q-polynomial with respect to θ_1, and (iii) the graph is a dual polar graph or a Hamming graph.

Reviewed by *Robert Beezer*

References

1. E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, London, 1984. MR0882540 (87m:05001)
2. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989. MR1002568 (90e:05001)
3. P. Cameron, Dual polar spaces, Geom. Dedicata 12 (1982) 75–85. MR0645040 (83g:51014)
4. Y. Egawa, Characterization of $H(n,q)$ by the parameters, J. Combin. Theory Ser. A 31 (1981) 108–125. MR0629586 (82k:05092)
5. A. Neumaier, Characterization of a class of distance-regular graphs, J. Reine Angew. Math. 357 (1985) 182–192. MR0783540 (86f:05109)
6. N. Sloane, An introduction to association schemes and coding theory, in: R. Askey (Ed.), Theory and Application of Special Functions, Academic Press, New York, 1975. MR0401326 (53 #5155)
7. D. Stanton, Some q-Krawtchouk polynomials on Chevalley groups, Amer. J. Math. 102 (4) (1980) 625–662. MR0584464 (82a:33015)
8. P. Terwillinger, Root systems and the Johnson and Hamming graphs, European J. Combin. 8 (1987) 73–102. MR0884067 (88d:05147)
9. P. Terwillinger, A new inequality for distance-regular graphs, Discrete Math. 137 (1995) 319–332. MR1312463 (95k:05187)
10. C. Weng, D-bounded distance-regular graphs, European J. Combin. 18 (1997) 211–229. MR1429246 (97m:05265)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2110581 (2005i:05088) 05C35 (05C20)
Baskoro, Edy Tri (IN-BIT); **Miller, Mirka** (5-BAL-ITM); **Širáň, Jozef** (SK-STU); **Sutton, Martin** (5-NEWC-DCI)
Complete characterization of almost Moore digraphs of degree three. (English summary)
J. Graph Theory **48** (2005), *no. 2,* 112–126.

A regular graph of degree r and diameter d may not have more than $\sum_{i=0}^{k} r^i$ vertices. This bound is known as the Moore bound, and graphs which meet it are known as Moore graphs. In the case of undirected graphs, these graphs are almost entirely known, the only case in question is the existence of graphs of degree 57 and diameter 2 on 3250 vertices.

For a directed graph, the degree condition is replaced by a common out-degree of r. Then the Moore bound is met only in the trivial cases when $r = 1$ or $d = 1$. However, there are interesting digraphs whose vertex count is just one shy of the Moore bound. These digraphs are called "almost Moore" and an infinite class of examples are the line graphs of complete digraphs.

As the title implies, this article is concerned with almost Moore digraphs for the case when $r = 3$; a classification for the case $r = 2$ was previously given by [E. T. Baskoro et al., Australas. J. Combin. **9** (1994), 291–306; MR1271209 (95e:05049); M. Miller and I. Friš, in *Graphs, matrices, and designs*, 269–278, Dekker, New York, 1993; MR1209196 (94a:05110)]. The authors' conclusion is that there are no almost Moore digraphs with $r = 3$ and $d \geq 3$. Two techniques are used frequently in this paper. The first is the repeat automorphism. Given a vertex v, find its three neighbors v_1, v_2 and v_3. For each v_i, draw the set of vertices corresponding to the trees of depth $d - 1$ below v_i. For a Moore digraph these three sets would be disjoint, but for an almost Moore digraph with $r = 3$, a single vertex will appear in two of the three sets. This unique vertex is the repeat of v, $r(v)$, and it happens that the map $r \to r(v)$ is an automorphism of the graph, known as the repeat automorphism. This automorphism is then used in the second technique, which the authors call a "Principle of Duality". This principle allows a theorem about a digraph G to translate to a theorem about the new digraph formed by reversing all the edges of G, provided the original theorem is stated in terms of arcs, neighborhoods, distances and the repeat automorphism.

This well-written paper concludes with the two obvious directions for further work: classify almost Moore digraphs with degrees greater than three, and study almost almost Moore digraphs (those whose vertex count misses the Moore bound by two).

Reviewed by *Robert Beezer*

References

1. E. Bannai and T. Ito, On finite Moore graphs, J Fac Sci Tokyo Univ 20 (1973), 191–208. MR0323615 (48 #1971)
2. E. Bannai and T. Ito, Regular graphs with excess one, Discrete Math 37 (1981), 147–158. MR0676421 (84d:05108)

3. E. T. Baskoro, On the existence of cubic digraphs with optimum order, MIHMI 3 (2) (1997), 27–34.
4. E. T. Baskoro, M. Miller, J. Plesník, and Š. Znám, Regular digraphs of diameter 2 and maximum order, Australasian J Combin 9 (1994), 291–306. MR1271209 (95e:05049)
5. E. T. Baskoro, M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to Moore bound, J Graph Theory 20 (1995), 339–349. MR1355433 (97e:05091)
6. E. T. Baskoro, M. Miller, and J. Plesník, On the structure of digraphs with order close to the Moore bound, Graphs and Combinatorics 14 (1998), 109–119. MR1628113 (99g:05085)
7. E. T. Baskoro, M. Miller, and J. Plesník, Further results on almost Moore digraphs, Ars Combinatoria 56 (2000), 43–63. MR1768596 (2001b:05101)
8. E. T. Baskoro, M. Miller, and J. Širáň, A note on almost Moore digraphs of degree three, Acta Math Univ Comenianae LXVI (2) (1997), 285–291. MR1620425 (99f:05042)
9. W. G. Bridges and S. Toueg, On the impossibility of directed Moore graphs, J. Combin Theory (B) 29 (1980), 339–341. MR0602426 (82f:05044)
10. R. M. Damerell, on Moore graphs, Proc Cambridge Philos Soc 74 (1973), 227–236. MR0318004 (47 #6553)
11. P. Erdős, S. Fajtlowicz, and A. J. Hoffman, Maximum degree in graphs of diameter 2, Networks 10 (1980), 87–90. MR0565328 (81b:05061)
12. A. J. Hoffman and R. R. Singleton, On Moore graphs with diameter 2 and 3, IBM J Res Develop 4 (1960), 497–504. MR0140437 (25 #3857)
13. M. Miller, Digraph covering and its application to two optimisation problems for digraphs, Australasian J Combin 3 (1991), 151–164. MR1122222 (93b:05081)
14. M. Miller, J. Gimbert, J. Širáň, and Slamin, Almost Moore digraphs are diregular, Discrete Math 218 (2000), 265–270. MR1754340 (2000m:05109)
15. M. Miller and I. Fris, Minimum diameter of diregular digraphs of degree 2, Computer J 31 (1988), 71–75. MR0932078 (89c:05042)
16. M. Miller and I. Fris, Maximum order digraphs for diameter 2 or degree 2, Pullman volume of Graphs and Matrices, Lecture Notes in Pure and Applied Mathematics 139 (1992), 269–278. MR1209196 (94a:05110)
17. M. Miller and J. Širáň, Digraphs of degree two which miss the Moore bound by two, Discrete Math 226 (2001), 269–280. MR1801075 (2001k:05099)
18. J. Plesník and Š. Znám, Strongly geodetic directed graphs, Acta F R N Univ Comen-Mathe XXIX (1974), 29–34. MR0363991 (51 #246)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2069049 (2005d:05153) 05E30 (05C50)
Fiala, Nick C. (1-OHS)
Strongly regular vertices and partially strongly regular graphs. (English summary)
Ars Combin. **72** (2004), 97–110.

To be "strongly regular" is a global property of a graph. A regular graph is strongly regular if, for each pair of distinct vertices x and y, the number of neighbors common to both x and y depends only on whether or not x and y are adjacent, and not on the particular choice of x and y themselves. It is easy to convert this definition into a local property of a vertex. Given a vertex x, examine every other vertex y. Should y be adjacent to x, count the number of vertices that are neighbors of both x and y, $\lambda(x,y)$. If y is not adjacent to x, count the number of vertices that are neighbors of both x and y, $\mu(x,y)$. If the values of $\lambda(x,y)$ and $\mu(x,y)$ are constant over all vertices y, then the author defines x as a "strongly regular" vertex. Not surprisingly, connecting these two ideas it is shown that a regular graph where every vertex is strongly regular will be a strongly regular graph.

This article picks off the low-hanging fruit related to this new definition in an entertaining and easily readable style, and also presents a few surprising results. Here is a sampling.

Suppose that x is a strongly regular vertex of a regular graph of degree k on v vertices, where λ and μ are the sizes of the sets of common neighbors relative to x. Then $k(k-\lambda-1)=(v-k-1)\mu$. (This is a familiar equation from the study of strongly regular graphs.)

Suppose that a regular graph on v vertices has $v-3$ or more strongly regular vertices, all with the same parameters, λ and μ. Then the graph is strongly regular. So while the proof is quite detailed, the author describes the result as "weak" since $v-3$ is so close to v. However, an extensive set of constructions in Section 3 provides arbitrarily large graphs with $v-4$ strongly regular vertices where the graph is not strongly regular.

It is known that a nontrivial regular graph is strongly regular if and only if it has at most three distinct eigenvalues. This paper shows that a connected graph with some, but not all, of its vertices strongly regular has at least five distinct eigenvalues.

The paper finishes with five conjectures and a table of the graphs on 10 or fewer vertices that have some strongly regular vertices, but that are not strongly regular graphs. The 31 graphs listed are described by the constructions of Section 3, or are simple combinations (unions, complements) of complete graphs, circuits or complete bipartite graphs.

Reviewed by *Robert Beezer*

References

1. A. E. Brouwer and J. H. van Lint, *Strongly regular graphs and partial geometries*, in: Enumeration and Design (Waterloo, Ont. 1982), Academic Press, Toronto, Ont., 1984, 85–122. MR0782310 (87c:05033)
2. P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge Univ. Press, Cambridge, 1991. MR1148891 (93c:05001)
3. M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, and J. Hemmeter, *Deza graphs: a generalization of strongly regular graphs*, J. Combin. Des. **7** (1999), 395–405. MR1711897 (2000i:05193)

4. C. D. Godsil and G. F. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001. MR1829620 (2002f:05002)
5. W. H. Haemers, *Eigenvalue Techniques in Design and Graph Theory*, Math. Centre Tract **121**, Mathematical Centre, Amsterdam, 1980. MR0568704 (81i:05003)
6. W. H. Haemers, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. **226/228** (1995), 593–616. MR1344588 (96e:05110)
7. J. J. Seidel, *Strongly regular graphs*, in: Surveys in Combinatorics (Proc. 7th Brit. Combin. Conf., Cambridge, 1979), pp. 157–180, London Math. Soc. Lecture Note Ser. **38**, Cambridge Univ. Press, Cambridge, 1979. MR0561309 (81c:05027)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2064865 (2005m:05092) 05C15 (05C05 94B05)
Hoory, Shlomo (IL-HEBR-IC); **Linial, Nathan** (IL-HEBR-IC)
Colorings of the d-regular infinite tree. (English summary)
J. Combin. Theory Ser. B **91** (2004), *no. 2,* 161–167.

This interesting article begins with a reminder about Moore's bound, the minimum number of vertices in a graph that is regular of degree d with girth r, denoted $n_0(d,g)$. It concludes with a discussion of error-correction codes, both perfect codes and linear codes. The principal device that connects these topics is the d-regular infinite tree, T_d.

If G is a finite graph on n vertices, regular of degree d with girth g, then if we view the graph as a one-dimensional complex, there is a cover map $\varphi\colon VT_d \to VG$. This function can be viewed as a coloring of T_d where each vertex of T_d is colored with the vertex of G that is its image under φ. This coloring has the property that if two different vertices of T_d have the same color, then they are at least a distance g apart in T_d.

So in the search for graphs meeting the Moore bound, we can first ask about colorings of T_d with n colors where similarly colored vertices are at least a distance g apart. The first (negative) result is that for any d and g there is a coloring of T_d with $n = n_0(d,g)$ colors. Moore graphs are relatively rare, so most of the colorings described by this result do not arise from cover maps of graphs. The authors denote a coloring that does arise from a graph as being "graphic" and give a necessary and sufficient condition for a coloring of T_d to be graphic. They then describe a coloring of T_d which comes close to being graphic, given that it is numerically close to meeting the terms of the equivalent condition. Here we want to keep similarly colored vertices far apart (at least distance g) but use as few colors as possible.

This last sentence should sound similar to the basic tension of designing an error-correcting code. Ensure that codewords (binary strings, the vertices of hypercubes) are very different (far apart in the corresponding graph) yet are also short strings (keep the underlying graph small). The vertices of T_d that are the preimage of a single vertex of G will form a distance-g code in T_d and the negative result above can be reinterpreted as positive: it says that there are codes in T_d, for any g, that are analogous to perfect codes for finite graphs. The article finishes with the statements of two theorems about subgroups of free groups and "linear" codes in T_d.

Reviewed by *Robert Beezer*

References

1. N. Alon, S. Hoory, N. Linial, The Moore bound for irregular graphs, Graphs Combin. 18 (1) (2002) 53–57. MR1892433 (2003b:05084)
2. N. Biggs, Algebraic Graph Theory, 2nd Edition, Cambridge University Press, Cambridge, 1993. MR1271140 (95h:05105)
3. N. Biggs, Constructions for cubic graphs with large girth, Electron. J. Combin. 5 (1) (1998) 25 Article 1 (electronic). MR1661181 (99j:05097)
4. R. Diestel, Graph Theory, 2nd Edition, Springer, New York, 2000. MR1743598
5. A. Lubotzky, R. Phillips, P. Sarnak, Ramanujan graphs, Combinatorica 8 (3) (1988) 261–277. MR0963118 (89m:05099)
6. J.H. van Lint, Introduction to Coding Theory, 3rd Edition, Springer, Berlin, 1999. MR1664228 (2000a:94001)

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MR1975940 (2004f:05194) 05E30 (05C75)
Hiraki, Akira (J-OSAKK3-DM)

A distance-regular graph with bipartite geodetically closed subgraphs. (English summary)

European J. Combin. **24** (2003), *no. 4,* 349–363.

For vertices u and v a distance i apart in a finite graph G, let $A_i(u, v)$ be the set of vertices at distance i from u and distance 1 from v. If the cardinality of this set is independent of the choice of the particular u and v, then this common value is denoted a_i. A portion of the defining property of a distance-regular graph is that a_i is defined for all sensible values of i. This paper is about distance-regular graphs where the initial values of the a_i are zero. In this case, if there is one instance of a particular subgraph construction that is bipartite and geodetically closed, then all similar such constructions have the same property. Loosely speaking, a subgraph is geodetically closed if every shortest path in the original graph that connects two vertices from the subgraph only passes through vertices of the subgraph. In other words, all shortest paths possible through the original graph stay within the subgraph.

Specifically, the main result of this paper applies to distance-regular graphs of diameter d with an integer t such that $2 \leq t \leq d-1$ and $a_i = 0$ whenever $1 \leq i \leq t-1$. For any pair of vertices x and y, define $\Pi(x, y)$ to be the subgraph induced by the set of vertices lying on shortest paths between x and y. Suppose there is a particular pair of vertices u and v a distance t or less apart such that $\Pi(u, v)$ is a bipartite geodetically closed subgraph. Then for any pair of vertices x and y a distance t or less apart the subgraph $\Pi(x, y)$ will be a bipartite geodetically closed graph. Moreover $\Pi(x, y)$ is either a path, an ordinary polygon, a hypercube or a projective incidence graph.

A second result provides two classifications of distance-regular graphs of diameter $d \geq 4$ with $a_{d-1} = 0$ where $\Pi(x, y)$ is a bipartite geodetically closed subgraph for any pair of vertices x and y. One of these classifications lists explicit possibilities for the graph as an ordinary polygon, a d-cube, folded $(d+1)$-cube, the odd graph O_{d+1} or doubled odd graph $2O_{\frac{d+1}{2}}$. In the course of proving these results, several related results of Koolen about when $\Pi(x, y)$ is a distance-regular graph are modified and reproved [see J. H. Koolen, J. Algebraic Combin. **1** (1992), no. 4, 353–362; MR1203682 (93k:05149)].

Reviewed by *Robert Beezer*

References

1. E. Bannai, T. Ito, Algebraic Combinatorics I, Benjamin-Cummings, California, 1984. MR0882540 (87m:05001)
2. N.L. Biggs, A.G. Boshier, J. Shawe-Taylor, Cubic distance-regular graphs, J. London Math. Soc. (2) 33 (1986) 385–394. MR0850954 (88d:05135)
3. A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, Heidelberg, 1989. MR1002568 (90e:05001)
4. A.E. Brouwer, J.H. Koolen, The distance-regular graphs of valency four, J. Algebraic Combin. 10 (1999) 5–24. MR1701280 (2000h:05236)
5. H. Cuypers, The dual Pasch's axiom, European J. Combin. 13 (1992) 15–31. MR1149177 (93e:51008)
6. Y. Egawa, Characterization of $H(n, q)$ by the parameters, J. Combin. Theory Ser. A 31 (1981) 108–125. MR0629586 (82k:05092)
7. A. Hiraki, A characterization of the doubled Grassmann graph, the doubled Odd graph and the Odd graph by strongly closed subgraphs, European J. Combin. 24 (2003) 161–171. MR1961556 (2004c:05221)
8. J.H. Koolen, A new conditions for distance-regular graphs, European J. Combin. 13 (1992) 63–64. MR1149181 (92k:05105)
9. J.H. Koolen, On subgraphs in distance-regular graphs, J. Algebraic Combin. 1 (1992) 353–362. MR1203682 (93k:05149)
10. J.H. Koolen, On uniformly geodetic graphs, Graphs Combin. 9 (1993) 325–333. MR1250511 (94m:05074)
11. D.K. Ray-Chaudhuri, A.P. Sprague, Characterization of projective incidence structures, Geom.

Dedicata 5 (1976) 361–376. MR0480101 (58 #300)
12. J. Rifá, L. Huguet, Classification of a class of distance-regular graphs via completely regular code, Discrete Appl. Math. 26 (1990) 289–300. MR1045033 (91d:05080)
13. C. Roos, A.J. Van Zanten, On the existence of certain distance-regular graphs, J. Combin. Theory Ser. B 33 (1982) 197–212. MR0693359 (84g:05101)
14. H. Suzuki, On strongly closed subgraphs of highly regular graphs, European J. Combin. 16 (1995) 197–220. MR1324430 (96m:05200)
15. P. Terwilliger, Distance-regular graph and (s, c, a, k)-graphs, J. Combin. Theory Ser. B 34 (1983) 151–164. MR0703600 (84i:05100)

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MR1953724 (2003i:05089) 05C50 (05C05)
He, Li [He, Li[3]] (1-MIT); **Liu, Xiangwei** (1-MIT); **Strang, Gilbert** (1-MIT)
Trees with Cantor eigenvalue distribution. (English summary)
Stud. Appl. Math. **110** (2003), *no. 2,* 123–138.

Construct a sequence of trees, T_r, recursively as follows. Fix a degree $k > 2$. T_1 has a central vertex of degree k that is adjacent to k additional vertices, each of degree 1. For each vertex, v, of degree 1 in T_r, add $k-1$ new vertices and edges joining these new vertices to v. This will form the next tree in the sequence, T_{r+1}. The graph T_r will have $k(k-1)^{r-1}$ "boundary" vertices of degree 1, and $1+k+k(k-1)+k(k-1)^2+\cdots+k(k-1)^{r-2} = \frac{k(k-1)^{r-1}-2}{k-2}$ "interior" vertices of degree k. Notice that T_r is a subgraph of the infinite homogeneous tree of degree k.

By exploiting the recursive structure of the adjacency matrix of these trees, the authors are able to precisely determine the majority of the eigenvalues and their multiplicities via the characteristic polynomial, in addition to analyzing the structure of the eigenvectors. With the exception of $r+1$ eigenvalues (a fraction that goes to zero for large r), each eigenvalue is shown to be of the form $\lambda = 2\sqrt{k-1}\cos\left(\frac{m\pi}{n+1}\right)$, $1 \le m \le n \le r$.

Subsequently, facts about the Euler totient function allow the multiplicities of these eigenvalues to be computed. For example, $\lambda = 0$ occurs with multiplicity $k(k-2)[(k-1)^{r-2}+(k-1)^{r-4}+(k-1)^{r-6}+\dots]$, which in the case of $k=3$ asymptotically approaches $\frac{1}{3}$ of all the eigenvalues. The eigenvalues $\lambda = 2\sqrt{k-1}\cos\left(\frac{\pi}{3}\right)$ and $\lambda = 2\sqrt{k-1}\cos\left(\frac{2\pi}{3}\right)$ each appear with a multiplicity that asymptotically approaches $\frac{1}{7}$ of all the eigenvalues in the case $k=3$. The symmetry of the eigenvalues about the origin (because the trees are bipartite) and the pattern of the multiplicities explain the use of the word "Cantor" in the article's title.

Some progress is made in describing the associated eigenvectors, with more success for the eigenvectors associated with the zero eigenvalue. The article concludes with a brief discussion of the effect on the spectrum as the result of three variants formed by adding new edges to the boundary vertices of T_r. In the first case, the boundary vertices with a common neighbor are all joined to each other to form a complete subgraph of degree $k-1$, in the second case a k-regular graph is formed by making k copies of T_r and identifying corresponding boundary vertices, and in the final case all of the boundary vertices are joined together in a big circuit forming an "outer loop".

Reviewed by *Robert Beezer*

References

1. I. Gutman, Characteristic and matching polynomials of some compound graphs, *Publ. Inst. Math. Beograd* 27:61–66 (1980). MR0621934 (82k:05093)
2. O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Commun. Math. Phys.* 25:190–232 (1972). MR0297280 (45 #6337)
3. A. Edelman, H. Eriksson, and G. Strang, The eigenvalues for a cycle plus a random cycle, in preparation.
4. S. Strogatz, Exploring complex networks, *Nature* 410:268–276 (2001).
5. D. Watts, *Small Worlds: The Dynamics of Networks between Order and Randomness*, Princeton University Press, Princeton, 1999. MR1716136 (2001a:91064)
6. D. Watts and S. Strogatz, Collective dynamics of 'small-world' networks, *Nature* 393:440–442 (1998).
7. C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York and London, 1993. MR1220704 (94e:05002)
8. A. Figa-Talamanca, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, Cambridge University Press, Cambridge, 1991. MR1152801 (93f:22004)
9. F. Chung, *Spectral Graph Theory, CBMS Conference Series 92*, American Math. Soc., Providence, 1997. MR1421568 (97k:58183)
10. C. R. Johnson and A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, *Linear Multilinear Algebra* 46:139–144 (1999). MR1712856 (2000e:05114)
11. B. McKay, On the spectral characterisation of trees, *Combinatoria* 3:219–232 (1977). MR0543669 (58 #27578)

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MR1997609 (2004f:05109) 05C50 (05C75)
Borovićanin, Bojana (YU-KRAS)
Line graphs with exactly two positive eigenvalues. (English summary)
Publ. Inst. Math. (Beograd) (N.S.) **72(86)** (2002), 39–47.

An old problem in algebraic graph theory is that of determining those graphs with just a few positive eigenvalues. So suppose that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the four largest eigenvalues of the adjacency matrix of a simple graph (for convenience, assume the graph has at least four vertices). The eigenvalues of any graph sum to zero, so $\lambda_1 = 0$ for only the null graph; for all other graphs, $\lambda_1 > 0$. J. H. Smith [in *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, 403–406, Gordon and Breach, New York, 1970; MR0266799 (42 #1702)] determined all those graphs for which $\lambda_1 > 0$ and $\lambda_2 \leq 0$. Graphs with $\lambda_3 < 0$ have been characterized, as well as the minimal graphs with both $\lambda_3 \geq 0$ and $\lambda_4 < 0$ (see below for a definition of minimal in this context). This paper is concerned with the question of graphs with both $\lambda_2 > 0$ and $\lambda_3 \leq 0$, i.e. those having exactly two positive eigenvalues. These graphs are known in the case of connected bipartite graphs [M. Petrović, "Contribution to the spectral theory of graphs" (Serbian), Ph.D. thesis, Univ. Beograd, Belgrade, 1984; per bibl.]; here the author considers the case of connected line graphs.

The author details a collection of graphs, $\mathcal{F}$, containing a specific graph on seven vertices, a specific graph on nine vertices, an infinite family indexed by one integer parameter and an infinite family indexed by three integer parameters. The primary result of this paper is that a connected line graph has exactly two positive eigenvalues if and only if it is an induced subgraph of $\mathcal{F}$ and it contains the path on four vertices or the triangle with a pendant edge as an induced subgraph.

Suppose P is some property of a graph. Then P is called hereditary if whenever a graph has property P, then every induced subgraph of the graph also has property P. A graph H is said to be forbidden for property P if whenever a graph G has H as an induced subgraph, then G does not have property P. Finally, a forbidden graph is called minimal for P if whenever a vertex is removed, the resulting subgraph has property P.

The second major result of this paper determines explicitly the line graphs that are minimal for the hereditary property $\lambda_3 > 0$. Twelve of these thirteen graphs have six vertices, the exception being the five-cycle.

Reviewed by *Robert Beezer*