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A New Generation of Mathematics Textbook Research and Development

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Abstract

This paper adopts a multimodal approach to the latest generation of digital mathematics textbooks (print and online) to investigate how the design, content, and features facilitate the construction of mathematical knowledge for teaching and learning purposes. The sequential organization of the print version is compared to the interactive format of the online version which foregrounds explanations and important mathematical content while simultaneously ensuring a high level of connectivity and coherence across hierarchical layers of mathematical knowledge. For example, mathematical content in the online version is linked to definitions, theorems, examples and exercises that can be viewed in the original context in which the material was presented, and the content can also be linked to mathematics software. Significantly, the development process for the new generation of mathematics textbooks involves using a 'design neutral' markup language so that the books are simultaneously published as both print books and online books. In this development process, the structure of the chapters, sections, and subsections with their various elements are explicitly marked-up in the master document and preserved in the output format, giving rise to new methodologies for large-scale analysis of mathematics textbooks and student use of these books. For example, tracking methodologies and interactive visualizations of student viewings of online mathematical textbooks are identified as new research directions for investigating how students engage with mathematics textbooks within and across different educational contexts.

Key Words

Online digital mathematics textbooks; multimodal approach to mathematics textbooks; knowl; large-scale mathematics textbook research; mathematics textbook data analytics; interactive visualizations

1. Introduction

The development and use of digital textbooks is identified as an emerging and significant area for the future direction of mathematics textbook research (Fan et al. 2013). This is an urgent research agenda, given that the relative strengths and weaknesses of digital mathematics textbooks are not yet fully understood, as Kilpatrick (2014) claims in an overview of the history of the evolution of mathematics textbooks. For example, interactive digital textbooks are seen to offer new opportunities for "participation, flexibility and personalization", which stand in sharp contrast to the authoritative stance of traditional texts books (Yerushalmy 2014, p. 13). However, as Yerushalmy (2014) explains, studies of how students use textbooks are very rare, and little is known about how teachers and students take advantage of the new opportunities afforded by interactive digital formats (Rezat 2013). In addition, digital formats raise concerns about authorship which is less transparent in multilinked online resources compared to print versions (Gueudet and Trouche 2012). The lack of transparency and problems associated with the conceptual navigation of non-sequential materials "imposes a new type of professional responsibility", where the structure of concepts and their interrelations need to be made visible in the design of digital textbooks (Yerushalmy 2014, p. 17). Pepin et al. (2017) share similar concerns, and state that there is an increasing need to understand the new spaces of connectivity and interaction that accompany the move from print to digital resources in mathematics education (Bates and Usiskin 2016). As Pepin et al. (2017, p. 645) claim, digital mathematics textbooks and other digital curriculum resources "offer opportunities for change" with regard to understanding the design and use of these educational materials, their quality, and the nature of teacher and student interactions with these resources (see also Bates and Usiskin 2016; Choppin and Borys 2017).

Given the current situation, Fan et al. (2013) call for research on the development process for textbooks (i.e. how textbooks are produced) and more advanced and sophisticated methodologies in mathematics textbook research that can be applied to large datasets, particularly in relation to the use of digital textbooks. These issues are addressed in this paper. First, a multimodal approach to the study of the latest generation of print and online mathematics textbooks is used to investigate how the design, content and features facilitate the construction of mathematical knowledge. Following this, it is shown how the development process for these mathematics textbooks, involving the use of a ‘design neutral’ markup language so the book can be published in print and online form, leads to new methodologies for large-scale analysis of mathematics textbooks and student use of these books. For example, data analytics and interactive visualizations of student viewings of online mathematical textbooks are identified as new research directions for investigating how students engage with mathematics textbooks within and across different educational contexts. In what follows, an overview of the different types of digital textbooks is provided in this section before moving to these three issues in mathematics textbook research: namely, new generation digital mathematics textbooks, the development process for these textbooks, and new methodologies for large-scale mathematics textbook research.

Different generations of digital textbooks can be characterized as having certain features, following Pepin et al. (2016). The first generation digital textbooks consist of digitized versions of the printed book, with fixed content and limited search and navigational features found in digital documents. However, the second generation is fundamentally different in terms of the digital book itself, offering new functionalities for interaction, personalized learning and access to online content, including supplements and continuously upgradable content. Pepin et al. (2016) explain how the second generation of digital textbooks is “associated with an essential change concerning *design*, *teacher agency*, and *authorship*” (p. 640, original emphasis). They identify three types or models of digital textbooks based on design features which relate to their characteristics and the ways they are used. The three models of digital textbooks are:

1. Integrative e-textbook: an “add-on” type model where the digital version of the traditional textbook is linked to other learning materials.
2. Evolving or “living” e-textbook: an accumulative, developing model where a core community has authored a digital textbook which undergoes continual development.
3. Interactive e-textbook: a “tool kit” model where the e-textbook, authored to function as an interactive textbook, is based on learning objectives that can be linked and combined. (Pepin et al. 2016, p. 640)

Pepin et al. (2016, p. 651) identify connectivity and mathematical coherence as important dimensions of the quality of a digital textbook:

> What does mathematical coherence mean, if the textbook offers a set of elements, to be used by the teacher? We argue here that, in this context, the notion of quality must be seen as different kinds of connections: connections between different mathematical topics, between old and new knowledge, between activities; connections between the e-textbook and other resources, within structured resources systems of the teachers; but also connections between the users and the authors of the textbook (Pepin et al. 2016)[1].

Pepin et al. (2016) establish two levels of connectivity in digital mathematics textbooks: the micro-level – that is, links across specific mathematical content and topic areas, different semiotic representations (e.g. algebraic and geometrical) and software applications; and the macro level – that is, interactions with other online resources, other users (e.g. teachers and students) and other materials. Given the crucial role which textbooks play in teaching and learning mathematics (e.g. Rezat 2009; Usiskin 2013), the relations between connectivity and mathematical coherence and the quality of the mathematics textbook relate to nature of mathematical knowledge itself. For this reason, mathematical knowledge is discussed before the latest generation of online mathematics textbooks is investigated.

2. Mathematical Knowledge: A Multimodal Discourse Approach

Following Bernstein (2000), knowledge is realized through two forms of discourse: horizontal discourse and vertical discourse. Horizontal discourse exists within the realm of the everyday knowledge and “it is likely to be oral, local, context dependent and specific, tacit, multi-layered and contradictory across but not within contexts” (Bernstein 2000, p. 157). A critical feature of horizontal discourse is that it is segmentally organized in terms of the sites or situational contexts where the discourse takes place. On the other hand, vertical discourse takes “the form of a coherent, explicit and systematically principled structure, hierarchically organized as in the sciences, or it takes the form of a series of specialized languages with specialized modes of interrogation and specialized criteria for the production and circulation of texts, as in the social sciences and humanities” (Bernstein 2000, p. 157). Vertical discourses are distributed through regulated processes of recontextualization, in contrast to the discrete, segmented forms of horizontal discourse which today are circulated via online platforms (e.g. social media, blogs).

Mathematics is a vertical discourse but it is not strictly hierarchical. As Bernstein (1999) explains, mathematics is "a horizontal knowledge structure as it consists of a set of discrete languages, for particular problems" (p. 164). There is hierarchy within discrete domains (such as Euclidean geometry or linear algebra) but not necessarily across them. Therefore, within the context of topic specific textbooks used at universities, it is legitimate to claim that the mathematical knowledge is hierarchical. In other types of mathematics textbooks, for example those generally used in primary and secondary schools, there is likely to be relatively strong hierarchy within a single chapter, but the ordering of chapters is not strictly hierarchical. Therefore, following Bernstein’s (1999, 2000) formulations, mathematical knowledge is realized through vertical discourse with a specific and coherent structure which is hierarchically organized within specific topics and domains.

Moreover, mathematics is a multimodal discourse, consisting of language, images and mathematical symbolism (e.g. Lemke 2003; O'Halloran 2015). The three semiotic resources (i.e. sign systems) have a specific role to play in the construction of mathematical knowledge: for example, language is used for introducing and explaining mathematical content, techniques, procedures and results; images provide a perceptual overview of mathematical concepts and relations; and the symbolism is the resource through which mathematical results are derived. Each of the three semiotic resources has an underlying grammatical organization for fulfilling these functions and in addition, these grammatical systems are designed to function in collaboration with each other so that shifts can be easily made across language, image and symbolism. The close collaboration between the three resources enables semantic expansions of meaning to take place in the construction of mathematical knowledge (O'Halloran 2015).

Mathematics discourse consists of different text types or genres (Martin 1997; Martin and Rose 2008); for example, research papers, lectures, textbooks and exams. Each genre achieves certain goals through characteristic configurations of linguistic, visual and symbolic elements that unfold in identifiable, patterned ways as structured multimodal texts. Genres consist of sub-genres (or mini-genres): for example, mathematics textbooks consist of chapters with subsections which contain definitions, theorems, explanations of theory, demonstration examples, practice examples and solutions. These sub-genres in turn consist of items: for example, the ‘Demonstration Problem’ in the mathematics textbook consists of the Problem and Solution, with associated elements: for example, the Problem consists of Statement of Problem Context, Questions (a)… (n) with sub-components (e.g. Tables, Diagrams, Graphs), and the Solution consists of Question Answers (a)…(n) with various sub-components (O'Halloran 2008). The identification of the genres and sub-genres provides a basis for understanding how mathematical knowledge is constructed using language, image and symbolism in mathematical textbooks and the potential difficulties for students learning mathematics (Rezat and Rezat 2017; Schleppegrell 2007).

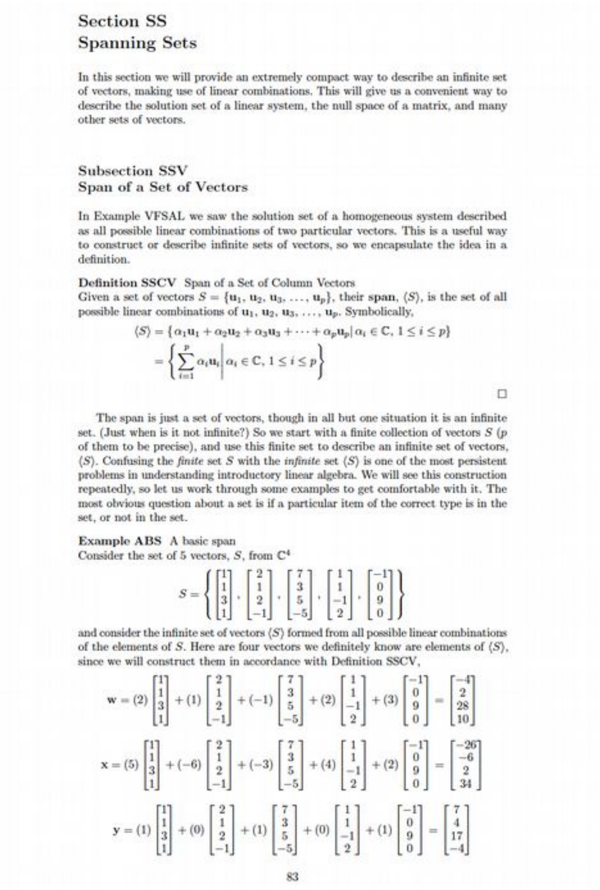
Given the hierarchical nature of mathematical knowledge and the explicit structure of mathematical textbooks, it is evident why the quality of digital mathematical textbooks is related to its connectivity. That is, mathematical coherence becomes a major issue in an online environment where mathematical content is linked to other content (the micro-level) and online resources and users (macro-level). In what follows, the latest generation of online mathematical textbooks at the tertiary level is investigated to explore the textbook design and the production process in terms of connectivity and mathematical coherence.

3. The New Generation of Online Digital Mathematics Textbooks

Robert Beezer’s book, *A First Course in Linear Algebra* (FCLA) is selected as an example of the latest generation of online mathematics textbooks. This textbook is chosen because it is written using a purpose-built ‘design neutral’ markup language so it can be simultaneously published as both a print book and an online book, as opposed to being written for one format and then translated into another format. In this sense, FCLA moves beyond the three types of digital textbooks which characterize the first and second generations of digital textbooks (see Pepin et al. 2016), and thus is selected for analysis.

FCLA is an introductory textbook designed for first and second year university students for a one semester course. The textbook is available in hard copy (Beezer 2015a), PDF (Beezer 2015b)[2] and online (Beezer 2017)[3]. Robert Beezer licenses the book with a GNU Free Documentation License (GFDL)[4] so that traditional copyright restrictions do not apply in this case. For this reason, FCLA is free in the sense that there is no cost to acquire it or to make copies, and users can modify copies of the book for personal use. However, if modifications are made and distributed, the person is required to apply the GFDL license to the result to ensure that others may benefit from the modification. A page from the print version (p. 83) and the corresponding screenshot of the online version of FCLA are displayed in Figure 1.

Beezer (2015a, b, p. ix; 2017[5]) explains that the online version is the most complete, integrating all the components of the textbook. We compare the print and online versions of the section from Beezer (2015b, 2017) displayed in Figure 1 in order to understand the similarities and differences between the two versions. As we shall soon see, features of the online version are designed to ensure connectivity and mathematical coherence, as well as providing the means for investigating how students interact with the textbook. Before examining the two books in detail, we first describe how both versions are possible.



Section SS
Spanning Sets

In this section we will provide an extremely compact way to describe an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the solution set of a linear system, the null space of a matrix, and many other sets of vectors.

Subsection SSV
Span of a Set of Vectors

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This is a useful way to construct or describe infinite sets of vectors, so we encapsulate the idea in a definition.

Definition SSCV Span of a Set of Column Vectors
Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \mid \alpha_i \in \mathbb{C},\ 1 \le i \le p\} = \left\{ \sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p \right\}$$

□

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite?) So we start with a finite collection of vectors S (p of them to be precise), and use this finite set to describe an infinite set of vectors, $\langle S \rangle$. Confusing the *finite* set S with the *infinite* set $\langle S \rangle$ is one of the most persistent problems in understanding introductory linear algebra. We will see this construction repeatedly, so let us work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not in the set.

Example ABS A basic span
Consider the set of 5 vectors, S, from $\mathbb{C}^4$

$$S = \left\{ \begin{bmatrix}1\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\end{bmatrix}, \begin{bmatrix}7\\3\\5\\-5\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\9\\0\end{bmatrix} \right\}$$

and consider the infinite set of vectors $\langle S \rangle$ formed from all possible linear combinations of the elements of S. Here are four vectors we definitely know are elements of $\langle S \rangle$, since we will construct them in accordance with Definition SSCV,

$$\mathbf{w} = (2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (2)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (3)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}-4\\2\\28\\10\end{bmatrix}$$

$$\mathbf{x} = (5)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (-6)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-3)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (4)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}-26\\-6\\2\\34\end{bmatrix}$$

$$\mathbf{y} = (1)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (1)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}7\\4\\17\\-4\end{bmatrix}$$

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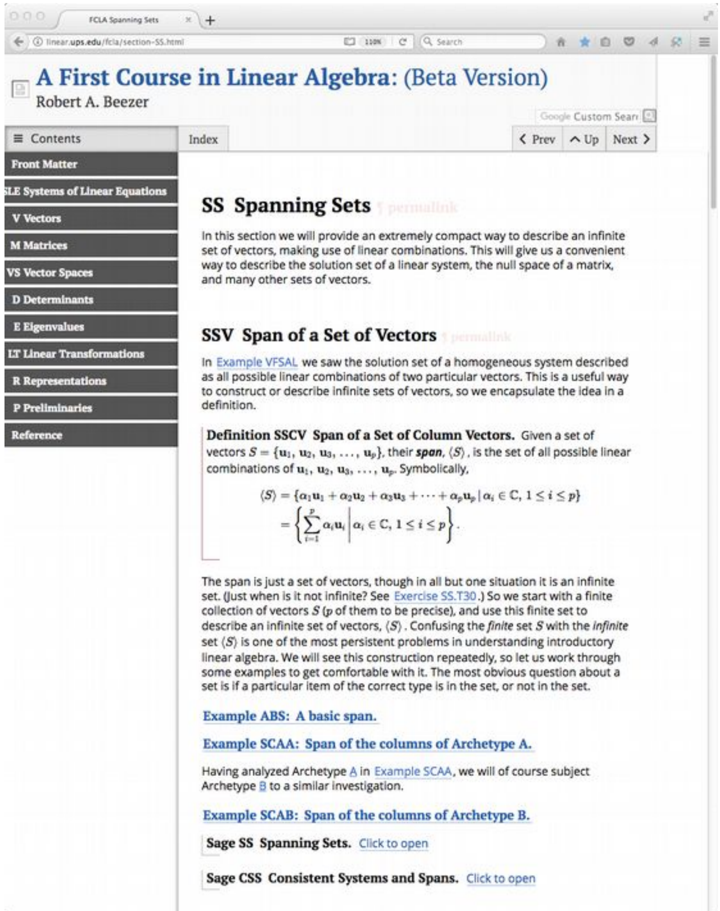
Figure 1a: Print Version (Beezer 2015a, p. 83)



A First Course in Linear Algebra: (Beta Version)
Robert A. Beezer

SS Spanning Sets

In this section we will provide an extremely compact way to describe an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the solution set of a linear system, the null space of a matrix, and many other sets of vectors.

SSV Span of a Set of Vectors

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This is a useful way to construct or describe infinite sets of vectors, so we encapsulate the idea in a definition.

Definition SSCV Span of a Set of Column Vectors. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their ***span***, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \mid \alpha_i \in \mathbb{C},\ 1 \le i \le p\} = \left\{ \sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\ 1 \le i \le p \right\}.$$

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite? See Exercise SS.T30.) So we start with a finite collection of vectors S (p of them to be precise), and use this finite set to describe an infinite set of vectors, $\langle S \rangle$. Confusing the *finite* set S with the *infinite* set $\langle S \rangle$ is one of the most persistent problems in understanding introductory linear algebra. We will see this construction repeatedly, so let us work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not in the set.

Example ABS: A basic span.

Example SCAA: Span of the columns of Archetype A.

Having analyzed Archetype A in Example SCAA, we will of course subject Archetype B to a similar investigation.

Example SCAB: Span of the columns of Archetype B.

Sage SS Spanning Sets. Click to open

Sage CSS Consistent Systems and Spans. Click to open

Figure 1b: Online Version (Beezer 2017)[6]

Figure 2: ***A First Course in Linear Algebra*** **(FCLA) by Robert Beezer (2015a, 2017)**

4. ‘Design Neutral’ Markup Language for Mathematics Textbooks

A First Course in Linear Algebra (FCLA) (Beezer 2015a, 2015b, 2017) is written using PreTeXt[7], an open source markup language for creating structured scholarly documents. When using this markup language, the author provides content and indicates the role or purpose of that content, but does not specify how that content is presented to the reader. A simple example is the title of a chapter. The author simply writes that the title of the chapter is: "A Textbook Research Study". PreTeXt converts those words into a form that a reader recognizes as a chapter title. For a print version of a document, the title might be left-justified, in a serif font, in black, at a 24 point font size. For an online version of the same document, the chapter title might be centered, in a sans serif font, in light blue, at a 30-pixel font size. Both versions of the title may be allocated a full stop, because it is not presumed that the author provides routine punctuation. It is possible to redefine these choices of style if required.

When modern word processors were first developed, they were described as "What You See Is What You Get" (WYSIWYG) in reaction to the perceived difficulties of the early text formatters which employed markup languages and computer programs for processing documents. However, there are many advantages to documents authored with a precise markup language. These advantages include:

- **Consistency in output**: for example, all chapter titles in a rendition of the document have an identical appearance.
- **Conversion to various output formats**: an author does not need to know anything about how the document will be rendered, and each rendering will faithfully preserve the organization and meaning of their content and present it in an optimal form (which can be adjusted if need be). Even for the choice of one physical output format, the same document can be rendered in many different ways or in different styles. For example, a PDF optimized to read on a laptop screen is different in many ways from a PDF sent as "camera-ready" copy to a publisher.
- **Authors concentrate on their specialty**: authors generally do not have content expertise or excellent skills in book design. Therefore, authors can devote their creative energy to writing about their chosen topic, without being distracted by instantaneous changes in the appearance of their final product. In this regard, the approach helps authors to organize their

writing in new and improved ways.

- **Structure is explicitly presented**: the markup language in a source document explicitly indicates the structure of the document, making it possible to investigate output in terms of this structure.

The explicit encoding of the structure of the document is significant in relation to mathematics textbooks, particularly as PreTeXt describes the document hierarchically. That is, a ‘book’ contains a sequence of ‘chapter’, and each ‘chapter’ is a sequence of ‘section’, and a ‘section’ may be further divided by a sequence of ‘subsection’. All of these divisions have a ‘title’. The author adds extra, explicit information by stating where each of these objects begins and ends, and then makes associations by placing objects within other objects. For example, every title will have the same begin and end markers, but each will be associated with a different chapter or section by being placed within the markers for its chapter or section. Furthermore, PreTeXt does not allow illogical constructions such as putting a chapter inside a subsection, or putting a subsection directly within a chapter without an intermediate containing section. A family tree is an apt analogy for how a PreTeXt document is structured. In this regard, the markup of a mathematical textbook is equivalent to a genre analysis where sub-genres and their elements are identified. As we shall see in Sections 6 and 7, marking-up the generic structure has significant implications for developing new methodologies involving automated analyses of mathematical textbooks and their use.

In terms of the textbook production process, content is specified in the following manner using PreTeXt. A ‘section’ would have a ‘title’ first, and then might be followed by several paragraphs, using begin and end markers named ‘p’ (for simplicity). Then the university-level mathematics textbook might have a ‘theorem’, delimited by begin and end markers of the same name. A theorem is typically a ‘statement’ of the result, followed by an optional ‘proof’ (or rarely, several different proofs). The author would use begin and end markers to reflect this structure. The ‘theorem’ can have an optional ‘title’. And ‘statement’ and ‘proof’ would be further structured as sequences of ‘p’ (paragraphs).

Within a ‘p’, an author can markup various constructions, such as a phrase to be emphasized or to be placed in quotes. The emphasis is a statement about the purpose of the linguistic elements, not what those words should look like. When processed, the emphasized words might be in italics, or in bold, or in bright red. Conversely, there is no markup in PreTeXt for font style, for example bold font. In this regard, it is important to consider what happens when an author uses a WYSIWYG word processor, highlights some text, and selects the “bold” formatting option. Significant information about the author’s intent has been lost by selection of a system choice for the style of the words, rather than the actual content. Moreover, bold has different meanings, depending on the context: for example, bold font is used for emphasis, a warning, a defined term, a journal volume number, and for arranging elements in a text (e.g. a heading). That is, when using word processing software such as Microsoft’s Word, the author is simultaneously the content expert, book designer and publisher.

In the case of mathematical symbolism, the inline expressions and displayed equations appear within the sentences of a paragraph using the syntax provided by TeX/LaTeX, which is popular and well-known mathematics writing software. This software permits the mathematical content to be expressed symbolically, where spatial layout, font size, special symbols, brackets and so forth, make specific meanings according to the specialized grammar of mathematical symbolism. For this reason, special typesetting software is required for mathematical expressions, in keeping the grammatical systems which permit meanings to be encoded unambiguously and economically (O'Halloran 2015).

Images are authored with graphics languages, so the image source can be embedded as part of the document source. This allows the automated conversion of the image to all available output formats (for examples of images in FCLA, see “DLTA: Definition of a Linear Transformation, Additive” and “DLTM: Definition of a Linear Transformation, Multiplicative”[8]).

In this sense, PreTeXt source has various provisions which make it possible to explicitly organize mathematical knowledge in a hierarchical fashion, while simultaneously incorporating the multimodal features (i.e. linguistic, visual and symbolic configurations) of the genres and sub-genres found in mathematical textbooks. In summary, these provisions are:

- Gross container structures such as ‘chapter’ and ‘theorem’.
- Narrative and content: paragraphs, titles, captions, block quotations, table cells, footnotes, etc. The PreTeXt schema makes this list very precise.
- Mathematical symbolism with its own (specialist) language, expressed with LaTeX syntax.
- Mathematical images, which preserve the original source inside the document, enabling the

automated conversion to any format.

When a textbook authored in PreTeXt is converted to an online version as a collection of web pages expressed in HTML, the structural information of the author's version is preserved in the output in a form which is invisible to the reader. That is, each chapter, section, theorem, example, exercise and solution is uniquely identified. As we discuss in Section 6, this codification facilitates the tracking of viewings through clicks on the interactive elements and scrolling through parts of the text. In other words, the requirement that an author explicitly describes structure while authoring (rather than post-production) allows that structure to be preserved and later related to the reader's physical interaction with the online textbook, which has significant implications for research on students' use of mathematical textbooks.

In what follows, the print and online versions of FCLA are discussed in order to illustrate how the mathematics textbooks appear to the reader and the differences between both versions. Following this, new methodologies for the automated large-scale analysis of mathematical textbooks and their use by students are presented.

5. Print Version and Online Versions

The generic structure of the print version of FCLA with Sections and Subsections consisting of Items (Section Title, Subsection Title, Introduction, Explanation) and sub-genres (Definitions, Example) is highlighted with overlays in Figure 2a. The figure is a screenshot from Multimodal Analysis Image[9] software, which has facilities for entering hierarchically organized systems and manually creating annotated overlays on images. The generic structure replicates the marking up of the document that has taken place using PreTeXt. In Figure 2a, however, this markup is highlighted in order to compare the print version with the online version in Figure 2b.

As seen in Figure 2a, the organization of mathematical content in the printed textbook is sequential with a clearly demarcated layout: that is, the Section consists of a Subsection, which in turn consists of Subsection Heading, Explanation, Definition, Explanation, and Example. In the case of FCLA, however, the parts are not numbered sequentially as found in traditional mathematics textbooks. Rather the sections and their elements are identified using acronyms, based on their name. This style realizes semantic content because the acronyms relate to the mathematical content. This is a feature of the way the book is authored using heavy cross-referencing and linking afforded by modern media, as exemplified in the discussion of the online version below.

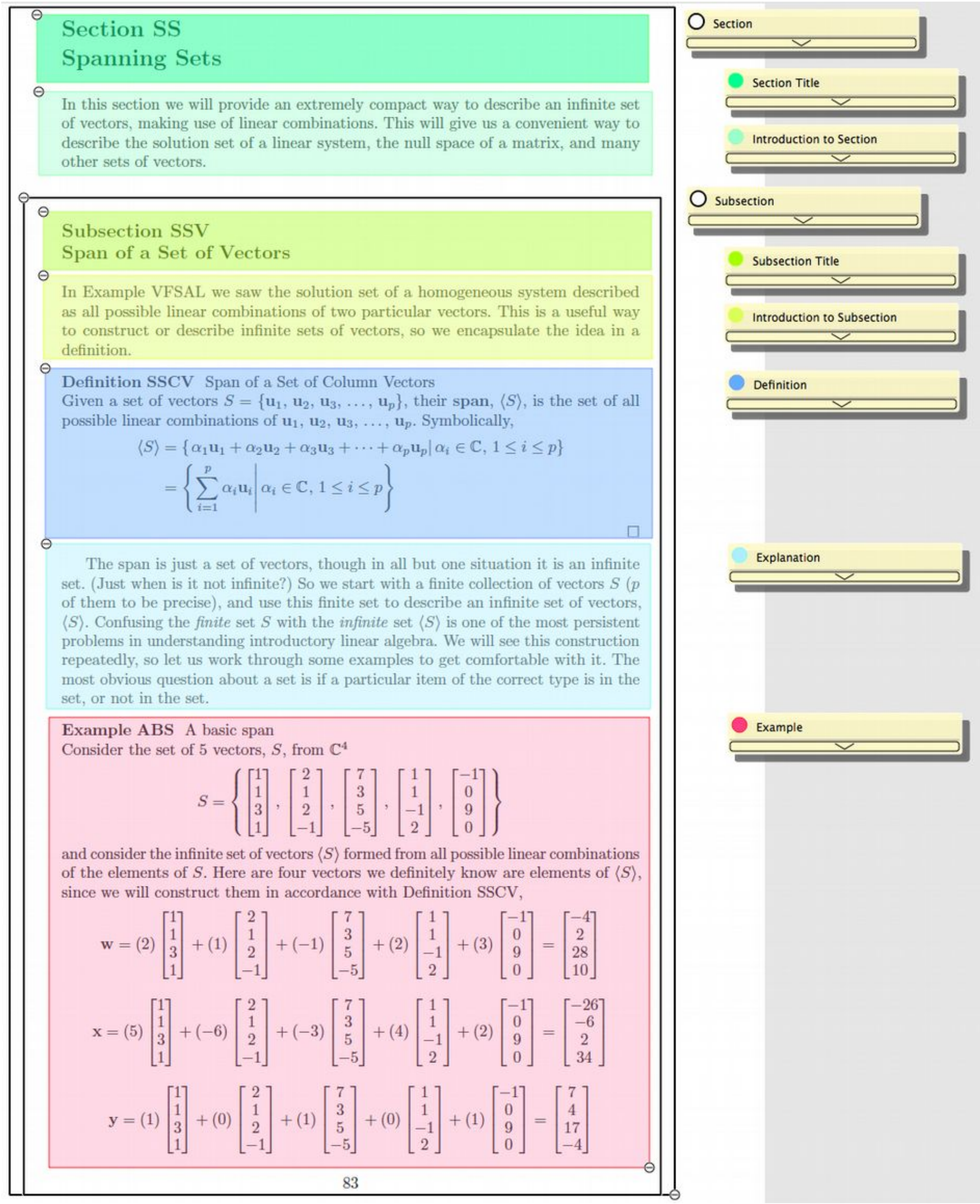


Figure 2a: Print Section Items (hard copy) (Beezer 2015a, p. 83)

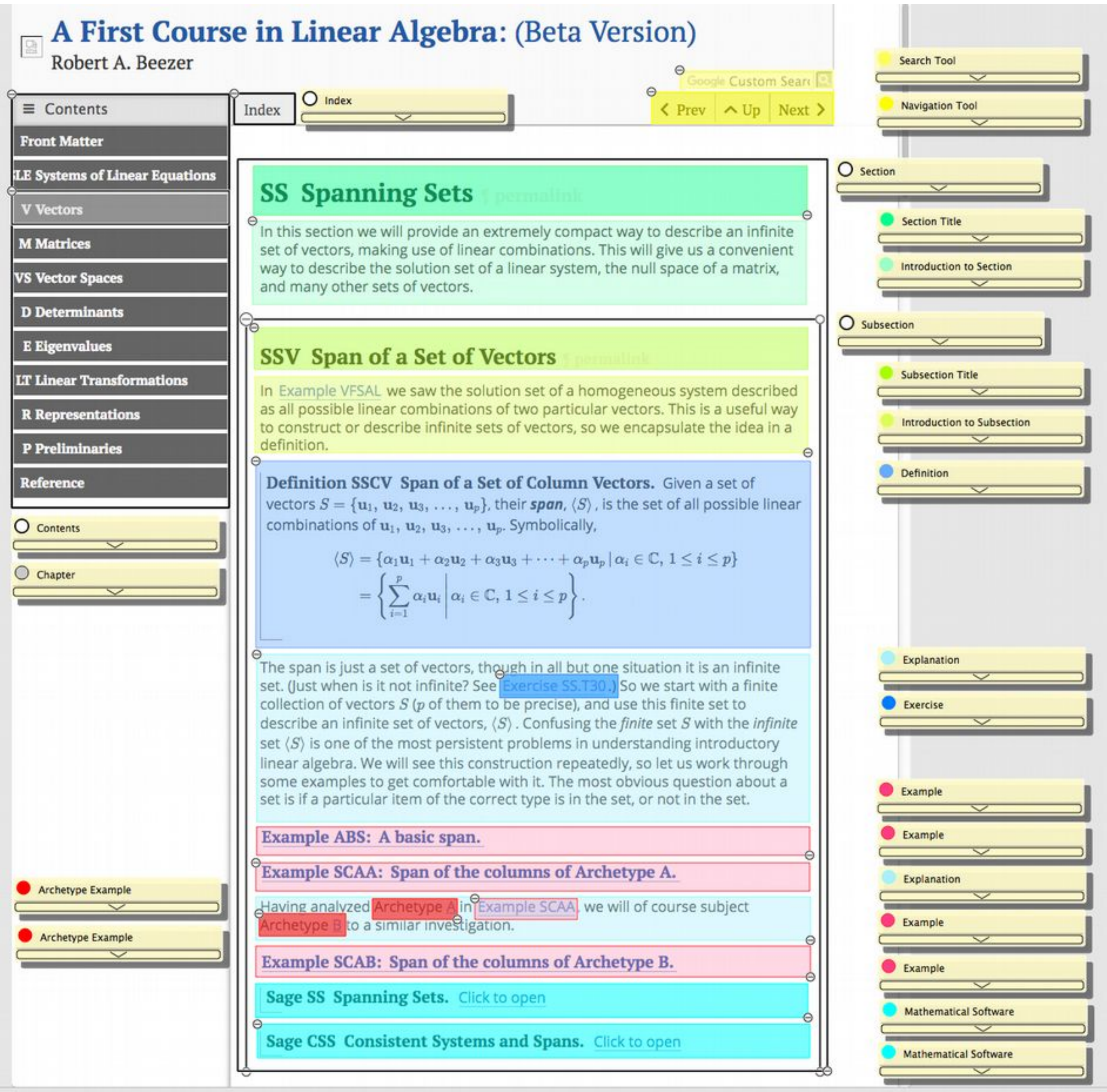
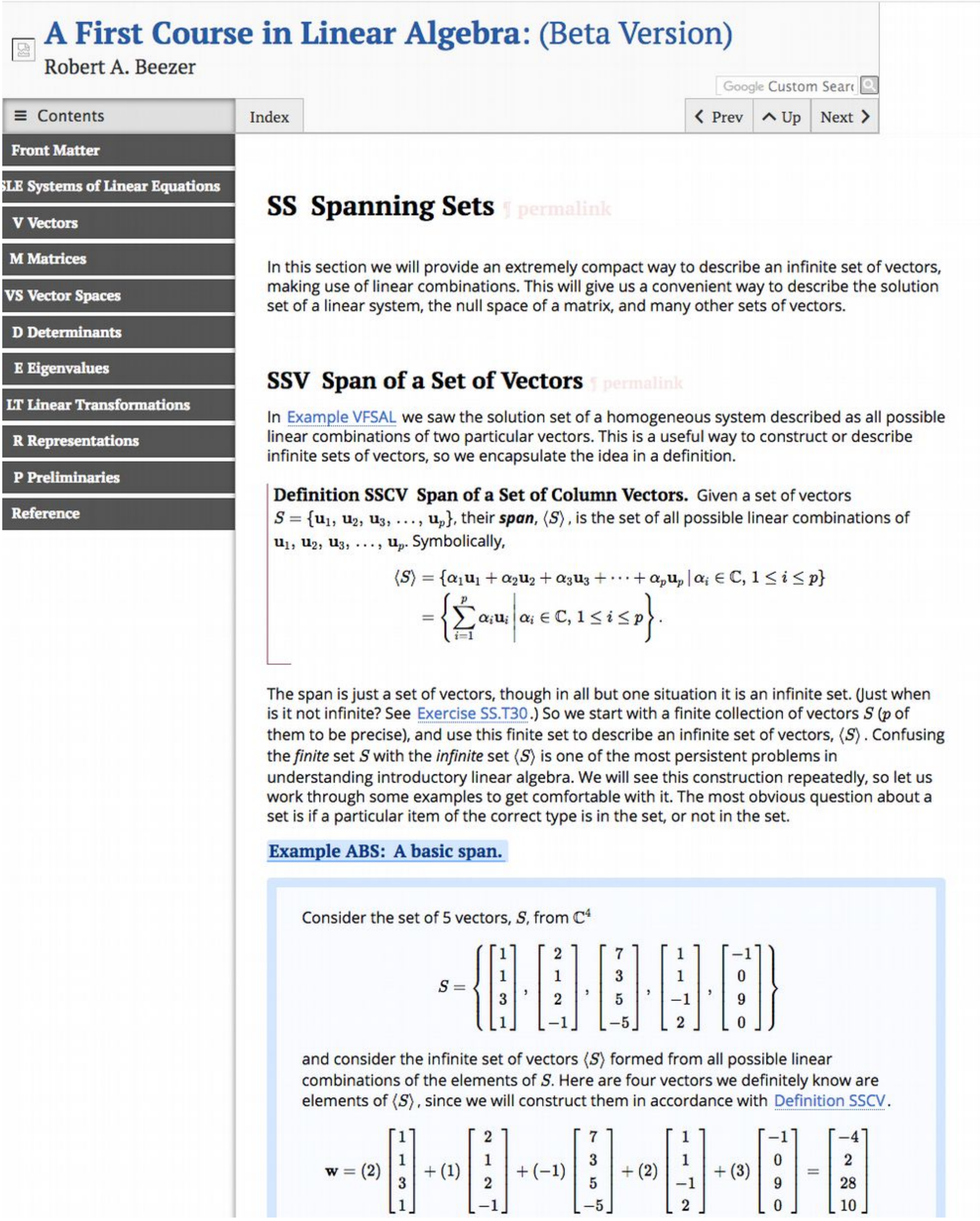


Figure 2b: Online Version Items (Beezer 2017)[6]

The online version displayed in Figure 2b replicates the organization of the mathematical content in the print version displayed in Figure 2a, but in addition, the online version contains further sub-genres. This includes Contents page with the list of Chapters (left panel), the Index (next to Contents title) and Navigational Tools (top right). While the actual mathematical content is replicated in the main panel in Figure 2b, we can see some significant differences, including a number of new elements. First, reference to an Exercise is embedded in the Explanation text with a dashed link. Second, three Examples are visible (instead of just one), with dashed links to each one. Third, the screenshot includes a further Explanation, with Archetype Examples (which are found in subsequent pages in the printed version). The Archetype Examples are a collection of twenty-four representative examples that serve as models for many of the theorems and definitions, and provide counterexamples to conjectures (Beezer, 2015a, 2015b, p. vii). Lastly, the online version contains a link to the software system SageMath[10]. SageMath is an open-source program with many tools for doing mathematics, including several devoted to linear algebra, with the result that linear algebra is one of SageMath's strengths. To the user, the SageMath calculations are embedded in the book, but the computations actually occur on a remote server.

The links with dashed lines in FCLA are called 'knowls'. When the knowls are clicked, boxes embedded in the text immediately below the dashed link are opened up, allowing the reader to access the content of definitions, theorems, examples, exercises, subsections and so forth, as displayed in Figure 3a. Knowls are also used for cross-referencing of the mathematical content, with a subsequent option to see this content "in-context" where it was originally presented. In addition, when a knowl with SageMath calculations is opened up, it is possible to read the code. The reader can choose not to open a knowl, in which case the text appears the same as in the print version. In this way, knowls permit considerable referencing of mathematical content, where the user can move through the various hierarchies of mathematical content at will, right down to the actual definitions upon which the results are obtained, as displayed in Figure 3b. This high level of connectivity of mathematical content across multiple levels is simply not possible in print versions of mathematical textbooks where such connections either remain implicit or consist of references only. Furthermore, printed books need to present materials in a concise sequential manner, given the production costs and the associated problem of limited space.

In this regard, it is evident that the new generation of online mathematics textbooks has the potential to create complex systems of interconnected elements that preserve the ordered and hierarchical structure of mathematical knowledge, while simultaneously making such knowledge coherent in terms of connecting the mathematical content across multiple layers. Significantly, the re-arrangement of the mathematical content in hierarchical layers brings language and the most significant mathematical content to the uppermost top level for the reader, as shown in Figure 2b. This form of foregrounding means that the mathematical content can be presented, explained and contextualized using language, while including important mathematical definitions and theorems. For example, Definition SSVC Span of a Set of Column Vectors appears alongside the written text in Figure 2b (see further examples in Figures 3a and 3b). From there, the reader can access proofs, exercises and examples as required, including the parts of the book where the material was first presented. In this regard, the textual organization of the mathematics textbook is streamlined in terms of explaining and presenting the mathematical content, while preserving the hierarchical organization of the mathematical content. The functionalities and facilities permit new forms of organization of mathematical content designed to ensure maximum coherence and connectivity.



A First Course in Linear Algebra: (Beta Version)

Robert A. Beezer

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SS Spanning Sets ¶ permalink

In this section we will provide an extremely compact way to describe an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the solution set of a linear system, the null space of a matrix, and many other sets of vectors.

SSV Span of a Set of Vectors ¶ permalink

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This is a useful way to construct or describe infinite sets of vectors, so we encapsulate the idea in a definition.

Definition SSCV Span of a Set of Column Vectors. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$, their ***span***, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S \rangle &= \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_p\mathbf{u}_p \,|\, \alpha_i \in \mathbb{C}, 1 \le i \le p\} \\ &= \left\{\sum_{i=1}^{p} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C}, 1 \le i \le p\right\}.\end{aligned}$$

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite? See Exercise SS.T30.) So we start with a finite collection of vectors S (p of them to be precise), and use this finite set to describe an infinite set of vectors, $\langle S \rangle$. Confusing the *finite* set S with the *infinite* set $\langle S \rangle$ is one of the most persistent problems in understanding introductory linear algebra. We will see this construction repeatedly, so let us work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not in the set.

Example ABS: A basic span.

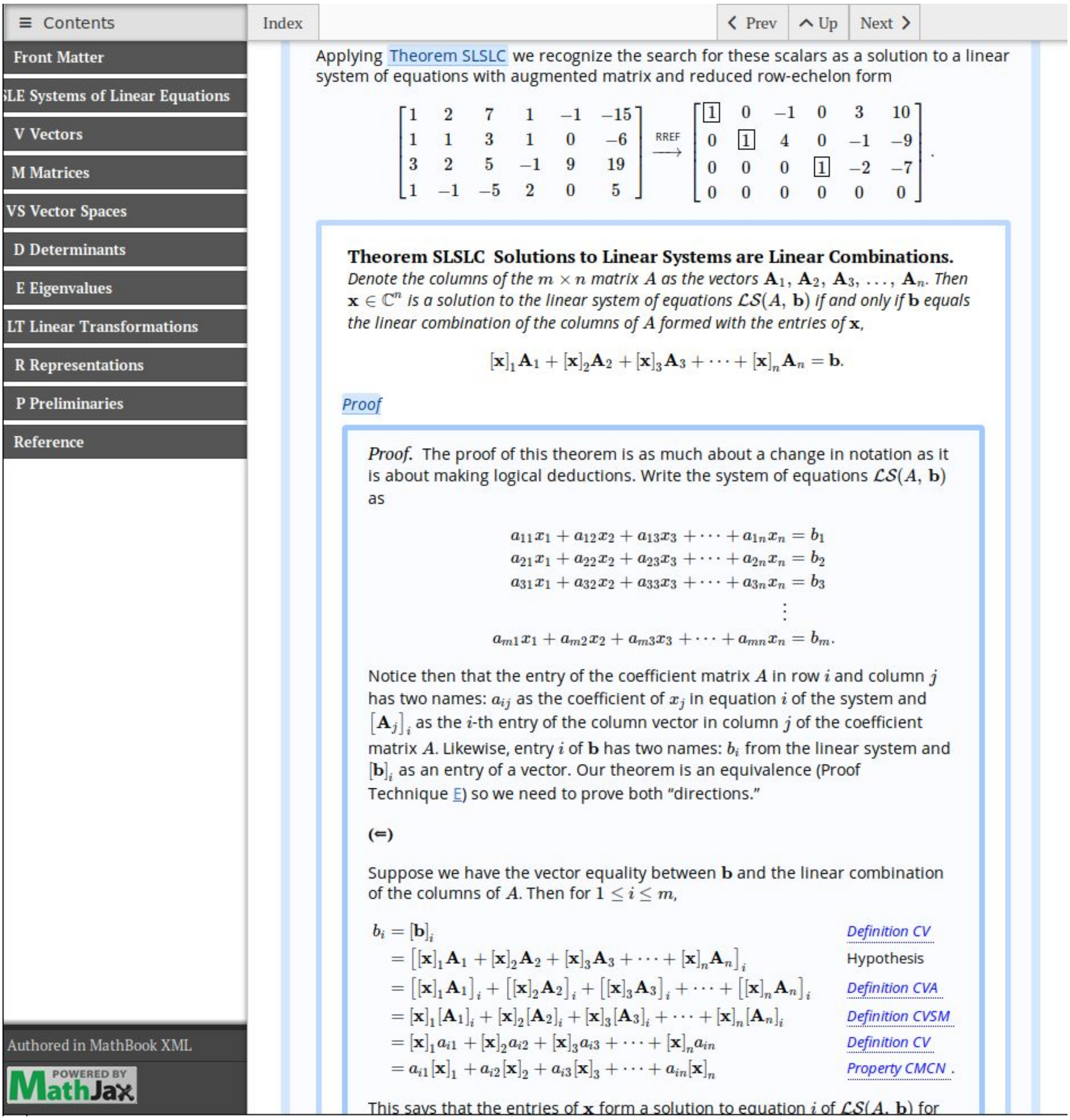
Consider the set of 5 vectors, S, from $\mathbb{C}^4$

$$S = \left\{ \begin{bmatrix}1\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\end{bmatrix}, \begin{bmatrix}7\\3\\5\\-5\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\9\\0\end{bmatrix} \right\}$$

and consider the infinite set of vectors $\langle S \rangle$ formed from all possible linear combinations of the elements of S. Here are four vectors we definitely know are elements of $\langle S \rangle$, since we will construct them in accordance with Definition SSCV.

$$\mathbf{w} = (2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (2)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (3)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}-4\\2\\28\\10\end{bmatrix}$$

Figure 3a: An Opened Knowl: Example ABS (Beezer 2017)[6]



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Applying Theorem SLSLC we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix and reduced row-echelon form

$$\begin{bmatrix} 1 & 2 & 7 & 1 & -1 & -15 \\ 1 & 1 & 3 & 1 & 0 & -6 \\ 3 & 2 & 5 & -1 & 9 & 19 \\ 1 & -1 & -5 & 2 & 0 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 3 & 10 \\ 0 & \boxed{1} & 4 & 0 & -1 & -9 \\ 0 & 0 & 0 & \boxed{1} & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem SLSLC Solutions to Linear Systems are Linear Combinations. *Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}^n$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{b}$ equals the linear combination of the columns of A formed with the entries of $\mathbf{x}$,*

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}.$$

Proof

Proof. The proof of this theorem is as much about a change in notation as it is about making logical deductions. Write the system of equations $\mathcal{LS}(A, \mathbf{b})$ as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Notice then that the entry of the coefficient matrix A in row i and column j has two names: a_{ij} as the coefficient of x_j in equation i of the system and $[\mathbf{A}_j]_i$ as the i-th entry of the column vector in column j of the coefficient matrix A. Likewise, entry i of $\mathbf{b}$ has two names: b_i from the linear system and $[\mathbf{b}]_i$ as an entry of a vector. Our theorem is an equivalence (Proof Technique E) so we need to prove both "directions."

(⇐)

Suppose we have the vector equality between $\mathbf{b}$ and the linear combination of the columns of A. Then for $1 \leq i \leq m$,

$$\begin{aligned} b_i &= [\mathbf{b}]_i && \text{Definition CV} \\ &= \left[[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n\right]_i && \text{Hypothesis} \\ &= \left[[\mathbf{x}]_1 \mathbf{A}_1\right]_i + \left[[\mathbf{x}]_2 \mathbf{A}_2\right]_i + \left[[\mathbf{x}]_3 \mathbf{A}_3\right]_i + \cdots + \left[[\mathbf{x}]_n \mathbf{A}_n\right]_i && \text{Definition CVA} \\ &= [\mathbf{x}]_1 [\mathbf{A}_1]_i + [\mathbf{x}]_2 [\mathbf{A}_2]_i + [\mathbf{x}]_3 [\mathbf{A}_3]_i + \cdots + [\mathbf{x}]_n [\mathbf{A}_n]_i && \text{Definition CVSM} \\ &= [\mathbf{x}]_1 a_{i1} + [\mathbf{x}]_2 a_{i2} + [\mathbf{x}]_3 a_{i3} + \cdots + [\mathbf{x}]_n a_{in} && \text{Definition CV} \\ &= a_{i1}[\mathbf{x}]_1 + a_{i2}[\mathbf{x}]_2 + a_{i3}[\mathbf{x}]_3 + \cdots + a_{in}[\mathbf{x}]_n && \text{Property CMCN}. \end{aligned}$$

This says that the entries of $\mathbf{x}$ form a solution to equation i of $\mathcal{LS}(A, \mathbf{b})$ for

Figure 3b: Embedded Knowls in Beezer (2017)[6]

The use of markup languages such as PreTeXt in the development process means that the structure of the mathematics textbook is recorded in terms of the sub-genres with their various elements (e.g. definitions, theorems, explanations of theory, demonstration examples, practice examples and solutions). The development process lends itself to data analytics of the textbook and of the student engagement with the textbook, as discussed in the following section.

6. Mathematics Textbook Analytics and Student Use of the Online Version

The marking up of mathematics textbook document, which equates to encoding the generic structure of the mathematics textbook, permits automated analysis of the textbook. For example, descriptions of the structure and the composition (i.e. the elements) of each chapter, section and subsection can be easily derived. In tandem, the markup of the mathematics textbooks makes it possible to obtain detailed information that can be used as a proxy for how the online textbook is being used by students (and also instructors) in different educational contexts over space and time (see the current study that is already underway in this area[11] (Mali and Mesa 2018, January; Mesa and Mali 2017, May)). The methodology for large-scale analysis of student use of online mathematics textbooks is described below.

The online book is viewed in a web browser (Firefox, Chrome, Safari, etc.), and so at any given moment a particular segment is visible. If "markers" are placed in the HTML source of the document, then in principle it is possible to determine which of those markers are visible at any specific time. Thus, it is possible to determine which portions of the book are visible at each moment with reasonable accuracy, and to record that information. The result is a precise description of each reader’s activity in viewing the book. The various dimensions of the data analytics for user viewings of FCLA are discussed below.

(a) What can be tracked?

In practice there are limitations to what can be tracked. Javascript running in the browser can be used to determine which markers have entered or left the viewing area. As we have seen, the chapters, sections, and subsections are completely analogous to those same items in the printed textbook. The online version of the book is organized as one HTML page per section, although a PreTeXt book can be arranged as desired (for example, a chapter, section, subsection can be one webpage). In the case of FCLA , one page naturally divides into subsections, and so markers are placed at the start of each subsection. As users scroll down a section, it is possible to detect when a subsection marker enters the top of the viewing screen, and this action is recorded.

It is possible to place numerous markers throughout each subsection and track whenever any marker enters the viewing window. However, this will lead to inaccuracies whenever more than one marker is in the viewing window, because it is not possible to know which is actually being observed by the user. Instead the focus is directed towards tracking interaction with specific features of the online book. For example, the examples, the proofs of the theorems, the solutions to problems, and the Sage cells, are not visible by default: the user must click to view those items. It is recorded when the viewer clicks to reveal each of those components, then it is recorded when they click again to close it, or click to view another component, or scroll to another part of the page.

(b) What is being tracked?

Javascript running in the browser is aware of all user actions and records any changes in viewing the page. At each moment there is a "current item in view". The current item changes whenever:

- The user goes to a new page. In this case, the current item is the section that is content of that page.
- The user scrolls and a new subsection and enters the viewing window. Now the current item is that subsection. Note that ‘reading questions’ and ‘exercises’ are considered subsections.
- The user clicks to view a knowl, proof, example, sage cell, or solution. Now the current item is the object that was clicked.
- The user scrolls while the current item is a knowl, proof, example, sage cell, or solution, or the user clicks to close a knowl, proof, example, sage cell, or solution. Now the current item is whatever subsection was most recently current.

(c) How is tracking recorded?

The tracking is recorded every second for the current item, and then the information is sent to the server every 10 seconds (or when the user leaves the page) and recorded in the server log. The log contains timestamped information about the IP address of the user, the operating system and browser, along with the information about the items viewed. The user is assigned a randomly generated reference number so their identity remains confidential at all times.

The data thus generated are large-scale, and for this reason, it is necessary to develop interactive visualizations to investigate how students and others (e.g. instructors) engage with the online mathematics textbook. In what follows, we present some prototype visualizations for displaying student viewing data of *A First Course in Linear Algebra* (FCLA).

7. Interactive Visualizations of Student Viewing Data

The interactive visualizations are designed to track student views of the online mathematics textbook over the course of a semester according to date and time. The aim is to display summaries for the whole class, and then to select areas of interest in relation to the parts of the textbook (e.g. chapter, section, subsection and element) and the time period. Of interest are the parts of the textbook that are most commonly and least viewed and the patterns of viewings over different time periods (the semester, the week, days, hours and minutes). In this way it is possible to situate the students' use of the mathematics textbook in relation to the actual mathematical content and the teaching and learning context (e.g. the course and the lectures, exams and study times) (Rezat 2009).

For instance, a screenshot of student viewing data for FCLA in a university course from January to April 2017 is displayed in Figure 4a. The vertical axis represents the chapters and sections of the online mathematics textbook and the horizontal axis is time. The heat map displays the sections of each chapter that are viewed each day according to student numbers (usage key: 0-50 students). Only the sections of FCLA that were actually viewed are displayed in Figure 4a. As can be seen, the pattern of viewings is quite marked, as evidenced by the diagonal arrangement of the heat map squares. This suggests that the students tend to view the chapters sequentially as they are covered in the course, with an increase in the number of student viewings during lectures and before exams, as indicated by the dark blue and grey heat map squares. It seems that students very rarely explore the content of the mathematics textbook outside the confines of how the material is presented in course materials and lectures, although some students certainly do this, as seen by the lighter squares underneath the main diagonal pattern in Figure 4a.

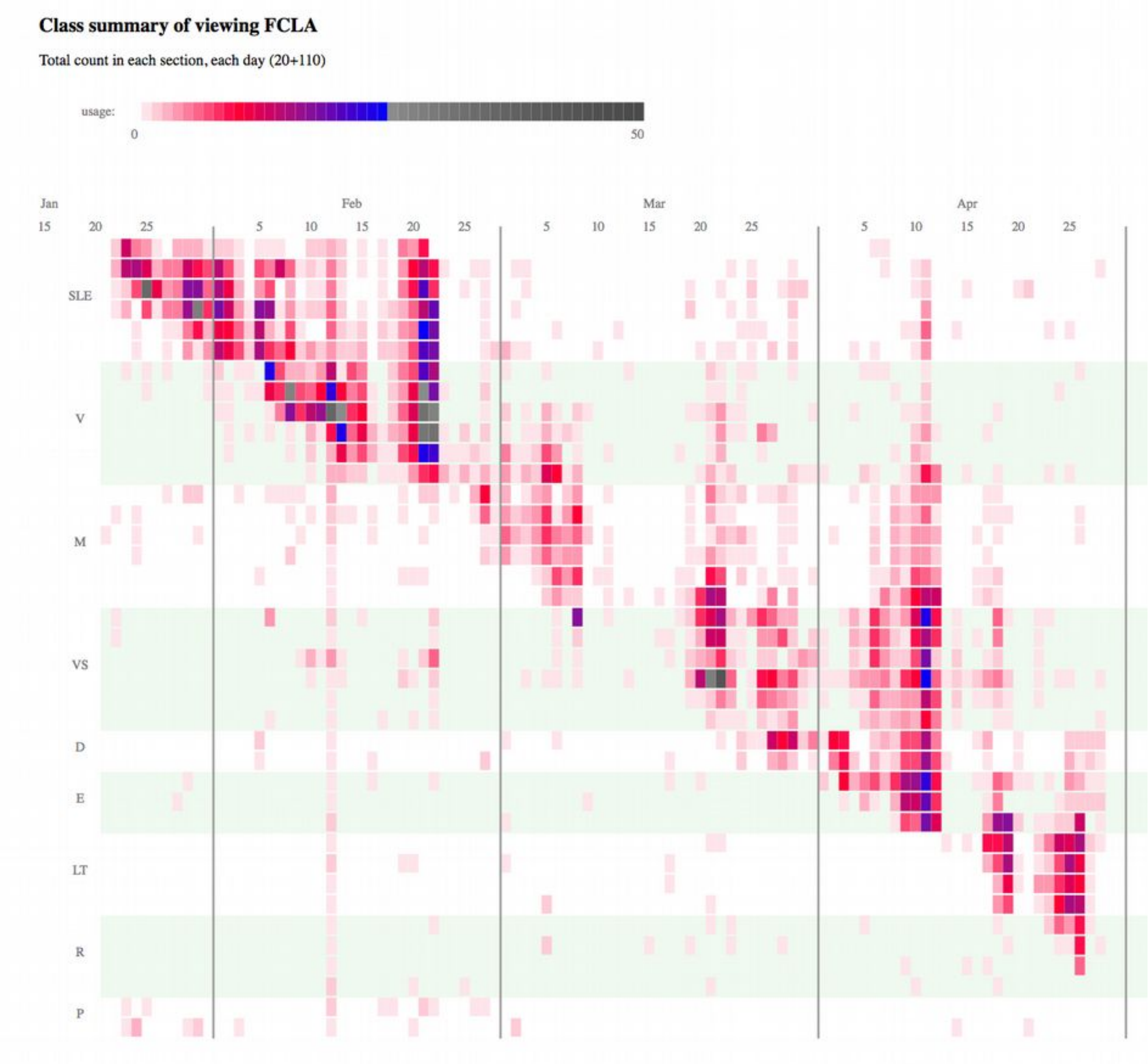

Figure 4a: Class Summary of Viewings of FCLA (January – April 2017)

The student viewings for each day are selected by hovering over the heat map displayed in Figure 4a and clicking on highlighted heat map squares for a particular day. For example, the student viewings for Tuesday 21 February 2017, the day before a class exam on Wednesday 22 February 2017, are displayed in Figure 4b. The heat map shows that students viewed the textbook from 5.00am to midnight on Tuesday and that the Vector (V) chapter was most commonly viewed, as indicated by the series of grey squares. The heat map for Wednesday 22 February 2017 (not shown here) shows that some students viewed the book from midnight to 10.00am (except from 3.00-4.00am) before the test took place at 11.00am to 12.00pm on that day. From here, it is possible to zoom into the viewing data for each chapter by hovering over and clicking on the corresponding highlighted heat map squares. The viewing data for the Vector (V) chapter on Tuesday 21 February is displayed in Figure 4c.

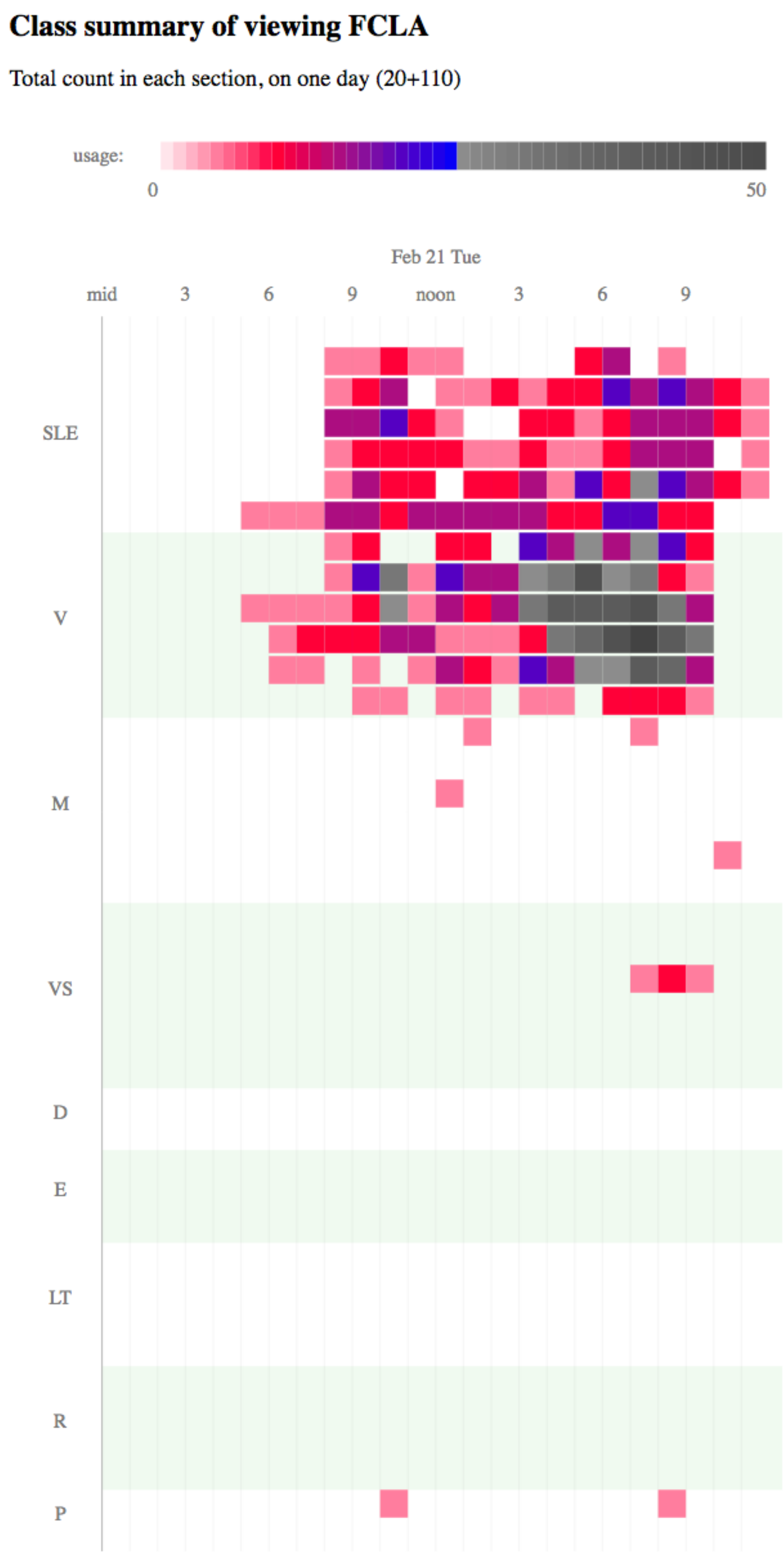

Figure 4b: Class Summary of Viewings of FCLA on Tuesday 21 February 2017

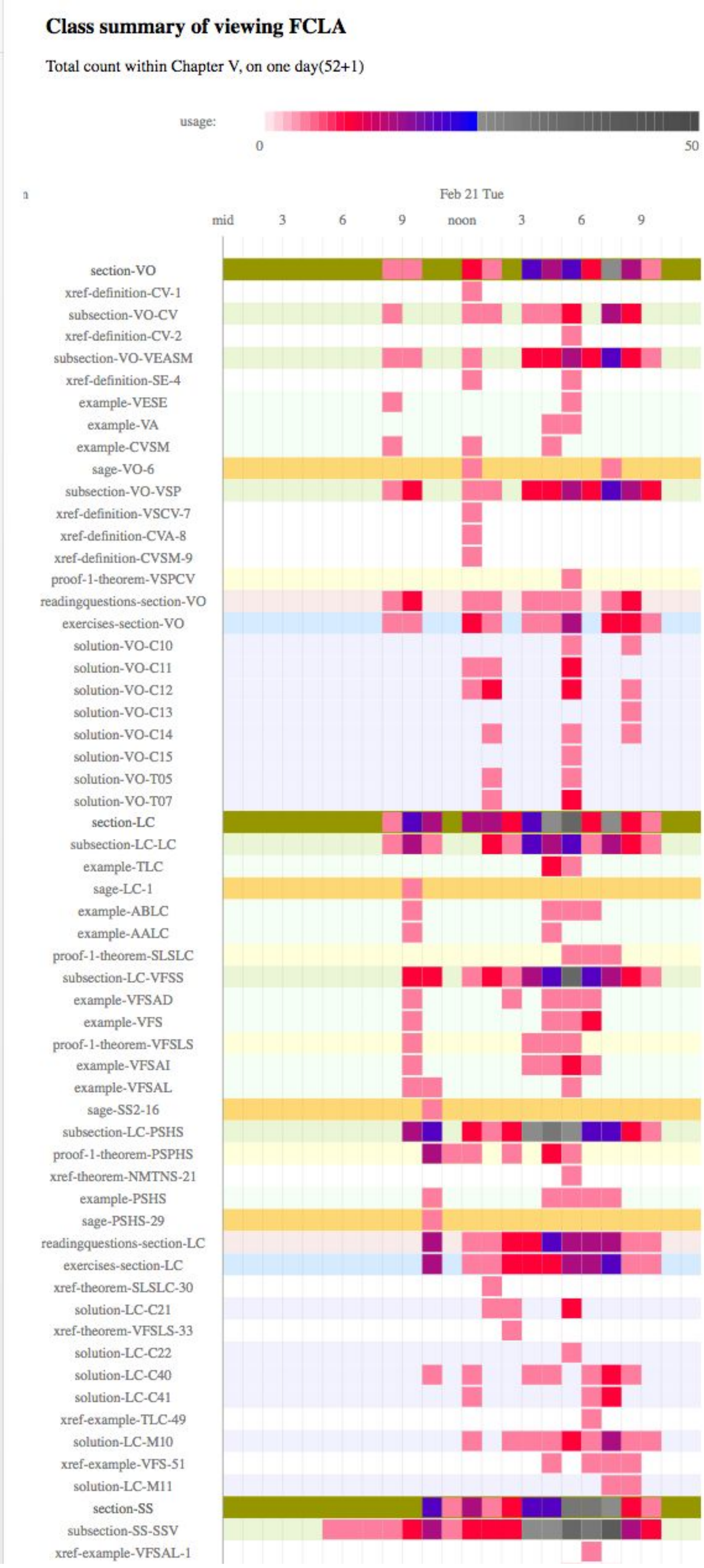

Figure 4c: Class Summary of Viewings of Vector (V) Chapter on Tuesday 21 February 2017

The student viewings per hour for the Vector (V) chapter are displayed in Figure 4c according to the section and subsection and the various items (e.g. definition, theorem, example, solution, and so forth) in the chapter. The grey squares of the heat map show that many students viewed the subsection Span of a Set of Vectors (SSV) in the section Spanning Sets (SS) for a relatively long period of time, indicating that this material was going to be tested in the exam and that it was potentially an area of difficulty for the students. However, as displayed in Figures 2a and 2b, there is much content in the Span of a Set of Vectors (SSV) subsection, including examples, definitions, exercises and Archetype examples that would have taken students time to work through.

The viewing data for individual students (per minute) can be accessed by hovering and clicking on the heat map squares for the class summaries displayed in Figure 4c. An example of individual student viewing patterns is displayed in Figure 4d. The vertical axis represents the part of the book being viewed and the horizontal axis is time (in minutes). In this case, the different coloured heat map squares represent individual students. It is possible to zoom into various parts of the heat map and select and display the data for a single student (not displayed here).

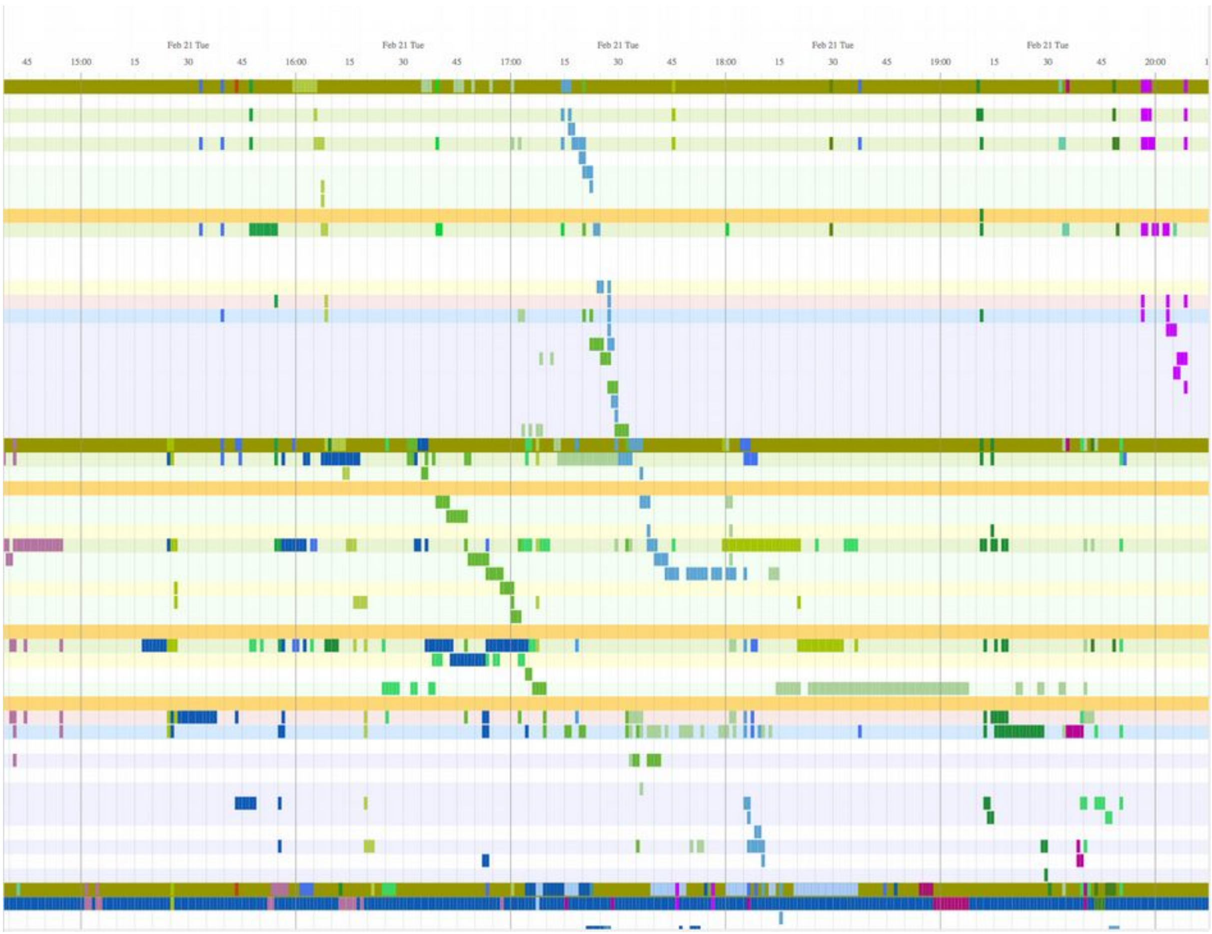

Figure 4d: Individual Viewings of Vector Chapter on Tuesday 21 February 2017 (per minute)

Lastly, the sum of the class viewings (in minutes) of items in each subsection, the subsections and sections over the whole semester can be viewed as a single bar chart of one semester of activity, as displayed in Figure 5, where total time for viewing has a logarithmic scale. The graph displays "bar" in the sense of "bar chart" so the width (not the area) is the important quantity for each item. The bar chart shows that the Vectors (V) chapter was viewed the most (not displayed in Figure 5) and within the chapter, the most viewed section was Spanning Sets (SS). Within that section, the most viewed subsection was Span of a Set of Vectors (SSV), which correlates with the information provided in the visualizations in Figures 4(a)-(d).

The viewing data raise interesting questions regarding the reasons why students view certain parts of the textbook and not others and the periods of time when they do this. Such questions can only be fully answered by correlating the student viewings with the teaching and learning context, as currently being undertaken in the study at the University of Michigan[11]. That is, the approach permits the data on textbook viewing to be correlated with the actual mathematical content and the context in which the material is being taught and learnt. Significantly, the methodology permits large datasets of mathematics textbooks and student viewing data to be studied over time. This approach provides the opportunity to study how student make use of the affordances for connectivity which the interactive digital textbook offers.

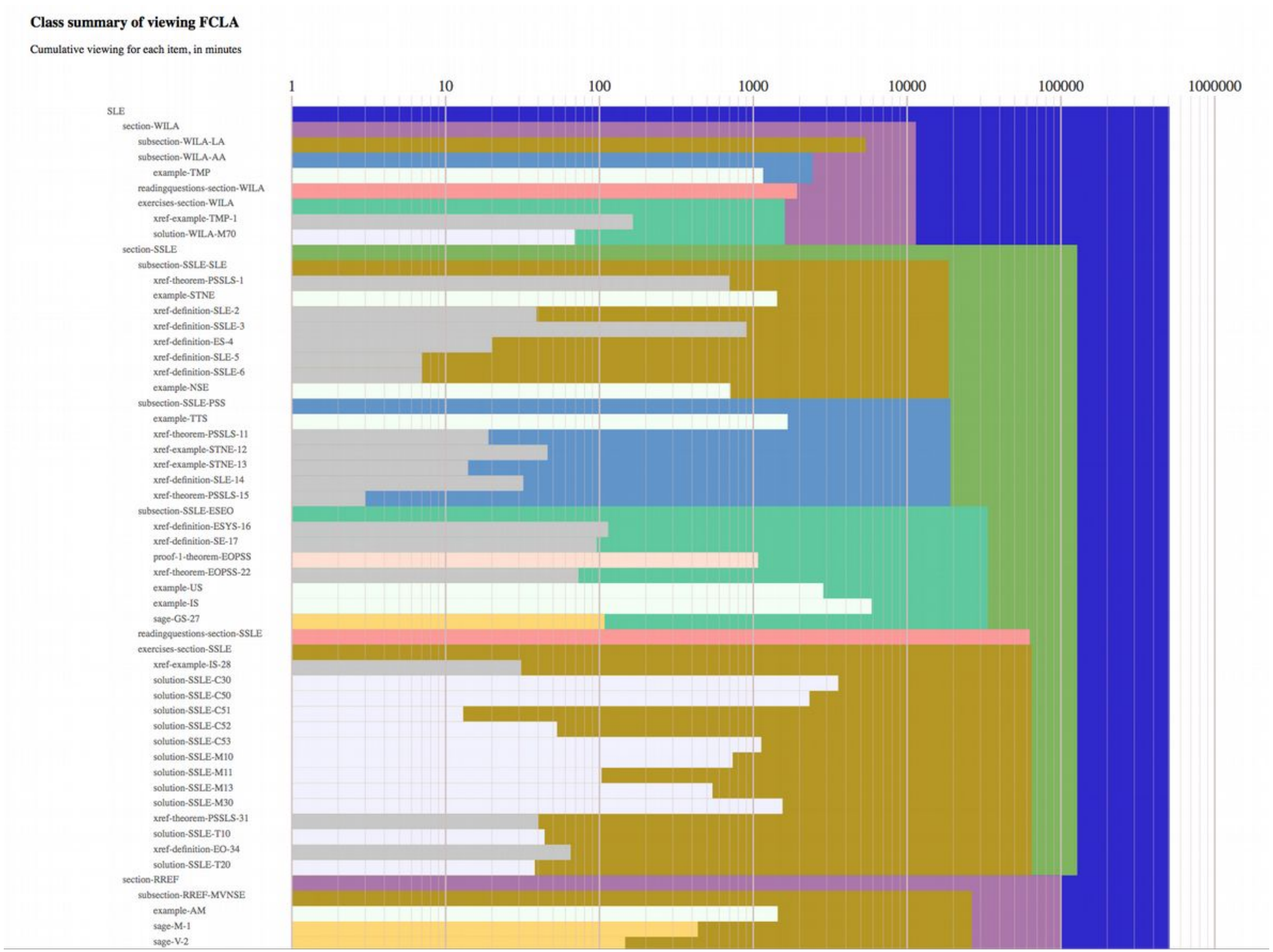

Figure 5: Cumulative Class Summary Figure of Viewings of FCLA (January – April 2017)

8. Conclusion

Mathematical knowledge has historically been constructed using different forms of representation, for example tally sticks, counters, writing implements, and paper. Modern mathematics developed as a written discourse involving language, image and mathematical symbolism. The three semiotic resources (i.e. sign systems) have each developed a sophisticated grammar which enables them to work closely together while fulfilling different functions in mathematics: that is, language is used to introduce and contextualize the mathematical content, images provide a perceptual overview of mathematical concepts and relations, and the symbolism is used to derive mathematical results. Today, online digital mathematics textbooks permit the complex hierarchical structure of mathematical knowledge to be presented and connected in new ways. For example, linguistic parts of the mathematics text can be foregrounded in order to introduce and explain mathematical concepts and to highlight important mathematical results that are visually and symbolically represented, while access via links to relevant definitions, theorems, examples, and exercises for this content is simultaneously provided. Moreover, the content can be connected to the original context in which the material was presented. In the hands of an expert author, such connectivity increases mathematical coherence, thus increasing the quality of the textbook for teaching and learning. The online digital environment also permits mathematical software to be incorporated as an integral part of mathematics today.

Moreover, the process of production, where mathematics textbooks are explicitly marked-up in terms of generic structure and compositional elements, has the potential to transform mathematics textbook research through the development of new research methodologies for large-scale analysis. These new methodologies include data analytics and interactive visualization approaches for studying the multimodal and semiotic structure of mathematical textbooks and exploring the ways students and others (e.g. instructors) use the books within and across different educational contexts. In particular, interactive visualizations permit patterns in large multidimensional data to be explored in new ways: for example, by overviewing the whole data set, zooming into areas of interest, selecting details on demand, extracting subsets of data, and so forth. Moreover, these visualizations are explicitly linked to parts of the mathematics textbook, so that mathematical content is directly related to student viewing patterns over space and time, which in turn can be related to the context (e.g. lecture, assignment, study period and examinations). Such investigations need to be informed by theories of mathematical knowledge and teaching and learning practices. While this study is limited to one textbook, the potential of the new generation of online digital textbooks for breaking new ground in terms of advancing mathematics textbook research and development is evident.

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Notes:

1. https://hal-emse.ccsd.cnrs.fr/CREAD/hal-01207678v1
2. FCLA PDF: http://linear.ups.edu/download/fcla-3.50-print.pdf
3. FCLA Online: http://linear.ups.edu/fcla/index.html
4. http://www.gnu.org/licenses/licenses.html#FDL
5. http://linear.ups.edu/fcla/preface-2.html
6. http://linear.ups.edu/fcla/section-SS.html

7. https://pretextbook.org

8. http://linear.ups.edu/fcla/section-LT.html

9. http://multimodal-analysis.com/products/multimodal-analysis-image/software/index.html

10. http://www.sagemath.org/

11. This work is part of a study being undertaken by Vilma Mesa, Angeliki Mali, Robert Beezer, and David Farmer at the University of Michigan. For further information, see: http://utmost.aimath.org/

Mathematics Subject Classification (MSC2010)

MSC Primary 97U20; Secondary 97U70, 97H60

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